Forms from Minkowski Triples of Circles

Daniela Velichová
Institute of Mathematics and Physics
Faculty of Mechanical Engineering, Slovak University of Technology
Nám. slobody 17, 812 31 Bratislava, Slovakia
E-mail: daniela.velichova@stuba.sk

Abstract

This paper deals with the modelling and study of various shapes arising from mathematical objects defined by means of Minkowski point set combinations. The aim is to present spatial objects with complex shapes generated by means of vector operations defined on smooth manifolds in Euclidean space. Examples using space curves that produce surfaces with unusual shapes are presented as flexible mathematical models suitable for various applications in computer aided modelling, design, architecture and art.

Introduction

Many different mathematical methods and approaches are utilized in the modeling of shapes. Basic principles include using geometric transformations; the movement of objects in space; the Boolean operations of union, intersection and difference, and interpolation and approximation. Minkowski point set operations offer a different approach to shape modeling. New objects can be determined from expressions involving the Minkowski sum and Minkowski product of two point sets. The resulting special configurations of points form various geometric structures of unusual forms with interesting features of symmetry and repeating motifs, or sometimes anti-symmetry, self-intersections, discontinuity and randomness. Configurations can be generated dynamically, using any available computer algebra system capable of visualizing the graphs of multivariable functions, and can serve as form inspiration for artists.

The Minkowski sum $\oplus$ and the Minkowski product $\otimes$ of two point sets are well defined algebraic operations on pairs of point sets in Euclidean (affine or projective) space of dimension n, (see [1]-[3]). Recall that the sum and product of two points a and b in $\boldsymbol{E}^n$ can be defined as the vector sum and outer vector product of the position vectors of the points a and b. These well defined vector operations are frequently used in abstract algebra. Let A and B be point sets in the space $\boldsymbol{E}^n$, $n \geq 2$.

Definition 1. The Minkowski sum $A \oplus B$ of the sets A and B is the set S consisting of all points in $\boldsymbol{E}^n$ that are the sum of a point a from A and a point b from B. That is, $S = A \oplus B = \{a + b \mid a \in A, b \in B\}$.

Definition 2. The Minkowski product $A \otimes B$ of the sets A and B is the set P consisting of all points in $\boldsymbol{E}^d$, $d = n(n - 1)/2$, that are product of a point a from A and a point b from B. That is, $P = A \otimes B = \{a \wedge b \mid a \in A, b \in B\}$.

These set operations are powerful tools for modelling unusual geometric shapes with added aesthetic value. By summing or multiplying two curves one can generate surfaces with unexpected shapes and properties. By introducing scaling parameters for the operand sets A and B we can produce smoothly deformable surface forms. Even more flexible mathematical models can be obtained by generalizations of the Minkowski operations leading to the concept of Minkowski point set combinations. These can be determined as images of Minkowski set operators, i.e. mappings defined on pairs of point sets in the potential set of the basic geometric space (see [4], [5]). Using three point sets, the concept of Minkowski triples can be introduced based on various combinations of the two Minkowski operations $\oplus$ and $\otimes$ for two point sets [6]. There are three different acceptable combinations available: the Minkowski *sum* triple, the Minkowski *product* triple, and Minkowski *mixed* triple. Together they offer a large variety of modelling possibilities.

Definition 3. Let A, B, C be point sets in the space $\boldsymbol{E}^n$ and k, l, h be real numbers. Define

$$W^{\oplus} = (k.A \oplus l.B) \oplus h.C = \{(ka + lb) + hc \mid a \in A, b \in B, c \in C\}$$

$$W^{\otimes} = (k.A \otimes l.B) \otimes h.C = \{(ka \wedge lb) \wedge hc \mid a \in A, b \in B, c \in C\}$$

$$W^{\oplus\otimes} = (k.A \oplus l.B) \otimes h.C = \{(ka + lb) \wedge hc \mid a \in A, b \in B, c \in C\}$$

Note that the Minkowski sum triple $W^{\oplus} \subset \boldsymbol{E}^d, d = n$, the Minkowski product triple $W^{\otimes} \subset \boldsymbol{E}^d, d = \dfrac{n(n^3 - 2n^2 - n + 2)}{8}$, and the Minkowski mixed triple $W^{\oplus\otimes} \subset \boldsymbol{E}^d, d = \dfrac{n(n-1)}{2}$.

The Minkowski triple of three equally parameterised curve segments is a curve segment, while a combination of three differently parameterized curves segments results in a 3-dimensional solid. A surface patch can be determined as a Minkowski triple of two equally parameterized curve segments $\mathbf{r}(u)$ and $\mathbf{s}(u)$, $u \in I$, and a third differently parameterized curve $\mathbf{q}(v)$, $v \in K$. Different combinations of parameterisations can be defined and the variety of resulting forms can be investigated. The coefficients k, l and m in the definition are shaping parameters. They permit dynamic smooth change of the generated manifold shape, size and form during the process of its modelling within the context of a computer algebra system.

Minkowski triples of circles

Consider three unique circles positioned in three perpendicular planes in the space $\boldsymbol{E}^3$. For example, the circles in the coordinate planes xy, yz, and xz are represented parametrically by the vector maps

$$\mathbf{r}(u) = (\cos u, \sin u, 0),\ \mathbf{s}(\mathrm{v}) = (0, \cos v, \sin v),\ \mathbf{q}(w) = (\cos w, 0, \sin w),\ u, v, w \in [0, 2\pi]$$

The parametric representation of the manifold determined as the Minkowski sum triple of these three circles can be expressed by the vector map

$$\mathbf{p}(u, v, w) = k\,\mathbf{r}(u) + l\,\mathbf{s}(v) + m\,\mathbf{q}(w) = (k \cos u + m \cos w, k \sin u + l \cos v, l \sin v + m \sin w),$$

where $(u, v, w) \in [0, 2\pi]^3$ and k, l and m are arbitrary real constants. These are the dynamic modelling parameters of a generated solid.

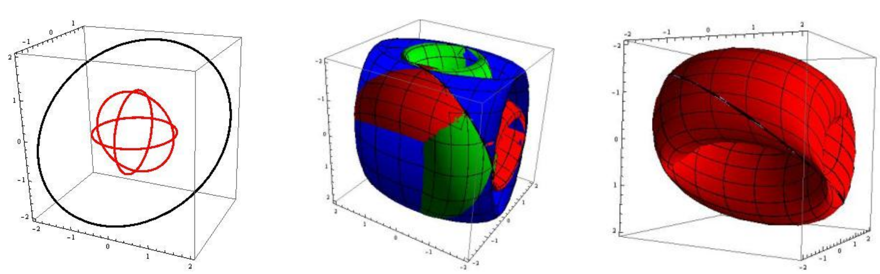

Figure 1: *Minkowski sum triples of three circles.*

The shape of the resulting manifold does not depend on the position of the three circles in their particular planes, but this influences its position in space. Three surface patches having the same form are produced for equal parameterisations $u = v$, $u = w$, or $v = w$, while the circle determined for $u = v = w$ is their common intersection as illustrated in Figure 1, which is interesting from a mathematical point of view.

More complex forms can be achieved by Minkowski product triples using the above three circles. In 3-dimensional Euclidean space $\boldsymbol{E}^3$ this will produce a manifold located in the same space $\boldsymbol{E}^3$, as can be concluded from the formula following Definition 3. The form of the resulting manifold now depends on the position of the circles in their particular planes. Its parametric representation for $(u, v, w) \in [0, 2\pi]^3$ is

$$\mathbf{p}(u, v, w) = klm(\cos u \sin v \sin w, \cos u \cos v \cos w - \sin u \sin v \sin w, \cos u \sin v \cos w),$$

where k, l and m are arbitrary real constants influencing the size of the resulting shape.
To produce a more complex dynamic model, 3 to 6 additional positioning parameters can be introduced, representing the coordinates of translation vectors in the three respective planes, enabling translation of basic circles. Examples of various forms of Minkowski product triple surfaces that were generated by changing their positions in the coordinate planes of the three circles are presented in Figure 2.

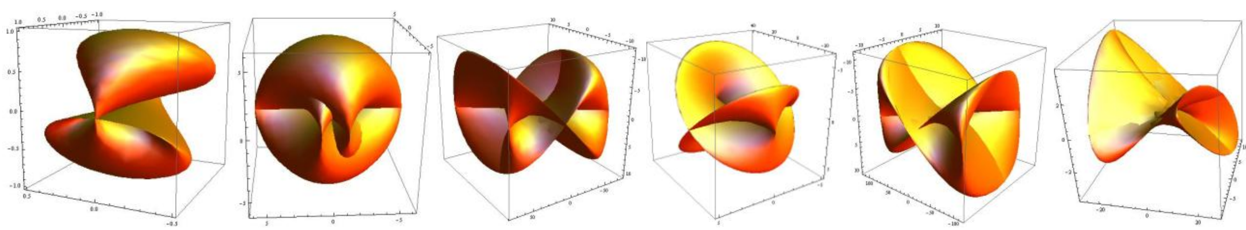

Figure 2: *Minkowski product triples of three circles.*

The most powerful tool for shape determination seems to be the Minkowski mixed triple. Three circles located in perpendicular planes generate as a Minkowski mixed product a manifold defined for $(u, v, w) \in [0, 2\pi]^3$ by

$$\mathbf{p}(u, v, w) = m(l \cos v \sin w + k \sin u \sin w, l \sin v \cos w - k \cos u \sin w, - l \cos v \cos w - k. \text{in}\ u \cos w).$$

Considering equal parameterisations of two mixed curves, the following configurations can be defined

$$\mathbf{p}(u, u, w) = (k\, \mathbf{r}(u) + l\, \mathbf{s}(u)) \wedge m\, \mathbf{q}(w),\ \ \mathbf{p}(u, v, u) = (k.\, \mathbf{r}(u) + l\, \mathbf{s}(v)) \wedge m\, \mathbf{q}(u), (u, w), (u, v) \in [0, 2\pi]^2 .$$

Examples of surface patches determined on $[0, 2\pi]^2$ by the first available form with vector map

$$\mathbf{p}(u, u, w) = m(l \cos u \sin w + k \sin u \sin w, l \sin u \cos w - k.\cos u \sin w, - l \cos u \cos w - k \sin u \cos u)$$

are presented in Figure 3.

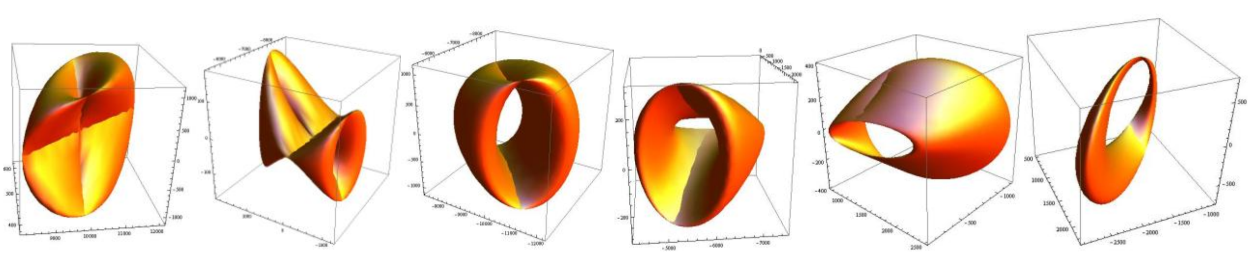

Figure 3: *Minkowski mixed triples of three circles, form* $\mathbf{p}(u,u,w)$.

The second form's surface patches illustrated in Figure 4 are determined on $[0, 2\pi]^2$ by the map

$$\mathbf{p}(u, v, u) = (l \cos v \sin u + k \sin^2 u, l. \sin v \cos u - k \cos u \sin u, - l \cos v \cos u - k \sin u \cos u).$$

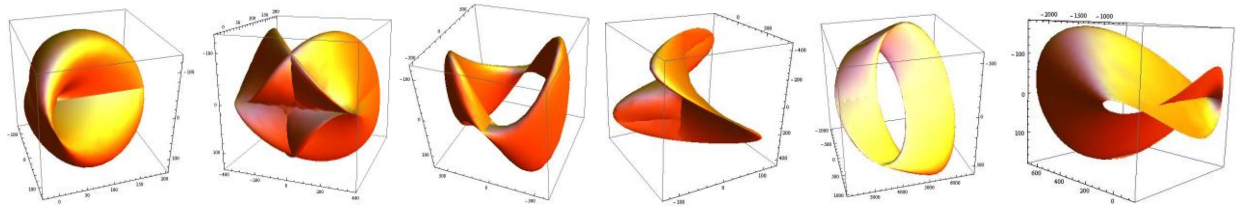

Figure 4: *Minkowski mixed triples of three circles, form* **p***(u, v, u).*

Modelling of manifolds in $\mathbf{E}^n$ is possible, and this presents a tool for shape definition in higher dimensional spaces. Such creative design of unusual forms of point sets in $\mathbf{E}^n$ can be used as a framework for modelling new shapes in $\mathbf{E}^3$ as orthographic views of these higher dimensional objects. Examples of surfaces generated as 3D orthographic views of Minkowski triples of three circles positioned in coordinate planes in the space $\mathbf{E}^4$ are presented in Figure 5.

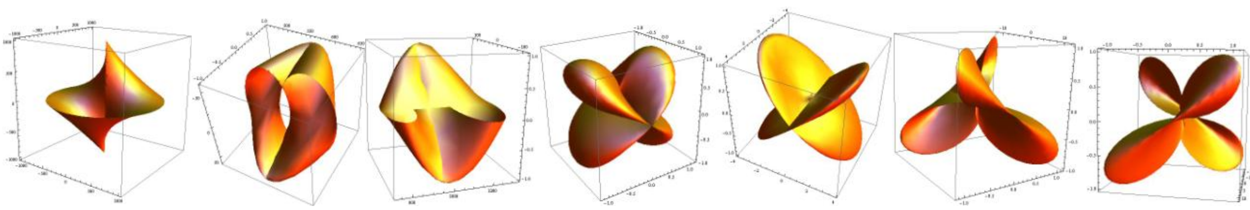

Figure 5: *Orthographic 3D views of Minkowski mixed triples of three circles from* $\mathbf{E}^4$.

Conclusions

Combinations of point sets determined as products of Minkowski point set operations are powerful mathematical tools for modelling forms that can be dynamically modified by changing various shaping parameters. It is hoped creative manipulations with these flexible objects in a dynamic software environment might inspire artists or designers to find among them a possible future sculpture or shape valuable from an aesthetic point of view. Because they are amenable to modifications, they might also serve computer artists as tools for generating metamorphoses of such shapes.

References

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