Nonspherical Bubble Clusters

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Abstract

Soap bubbles have always captured the imagination of artists as well as of children. We present computer graphics renderings of some small bubble clusters of mathematical interest. A single soap bubble is a perfectly round sphere; it seems that the soap films in (stable) clusters of small numbers of bubbles are always pieces of spheres. We focus on a cluster of six bubbles where this is not the case – in particular its central film is a saddle-shaped minimal surface. My computer-graphics rendering of this cluster dates from 1990. After it was featured last year in Ziegler's new book of mathematical pictures, I returned to it, printing it for exhibition for the first time and describing it here.

Children love to play blowing soap bubbles; over the centuries painters and other artists have often captured such moments [3]. While a single bubble always takes a perfectly round, spherical shape, when several bubbles stick together in a cluster, their shapes can vary. Mathematically, we can attempt to model the shapes of bubble clusters as solutions to a minimization problem. Here each bubble encloses a fixed amount of air, which we can pretend has fixed volume. The bubble cluster problem then asks for the least area way to enclose and separate k regions in space with prescribed volumes $V_1, \ldots, V_k$. I have previously explored the extent to which such geometric optimization problems lead to aesthetically pleasing shapes [7, 8].

Frank Morgan gives a good survey [6] of known theoretical results, which we summarize here. Almgren and Taylor showed that a least area cluster always exists and follows the rules first observed by Plateau almost exactly one hundred years earlier: Each film is a smooth surface of constant mean curvature (CMC); these films meet in threes at equal angles along smooth curves called Plateau borders; these curves in turn meet in fours at tetrahedral angles. Stated more simply, that means that (under a magnifying glass) each singularity looks like that formed by dipping a tetrahedral wire frame into soapy water, as in Figure 1. This abstract existence theory cannot ensure that each bubble in a minimzing cluster is connected – we must consider the possibility that it might be best to split one of the given volumes among two or more components.

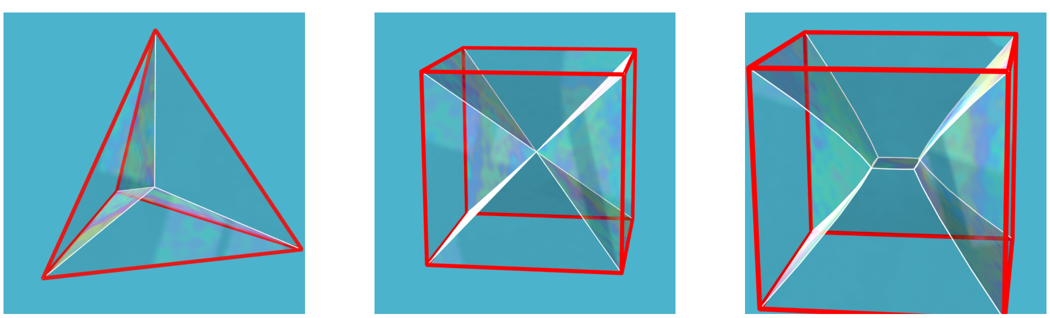

Figure 1 : *The soap film (left) spanning a tetrahedral wire frame exhibits the singularities allowed by Plateau's rules: Plateau borders meeting tetrahedrally. The film spanning a cube does not form a new singularity (center); instead this breaks apart into four of the allowed singularities (right).*

For a single bubble ($k = 1$) of course the solution is a round sphere; although Greek mathematicians including Archimedes understood this in some sense over 2000 years ago, a rigorous proof had to wait for Schwartz in 1884. In a double bubble ($k = 2$) the outer surfaces of both bubbles are spherical caps meeting along a circular Plateau border; since the smaller bubble has higher pressure, it pushes into the larger bubble and the inner film between them is also a spherical cap, as in Figure 2. (In the case of equal-size bubbles, this inner film is a flat disk.)

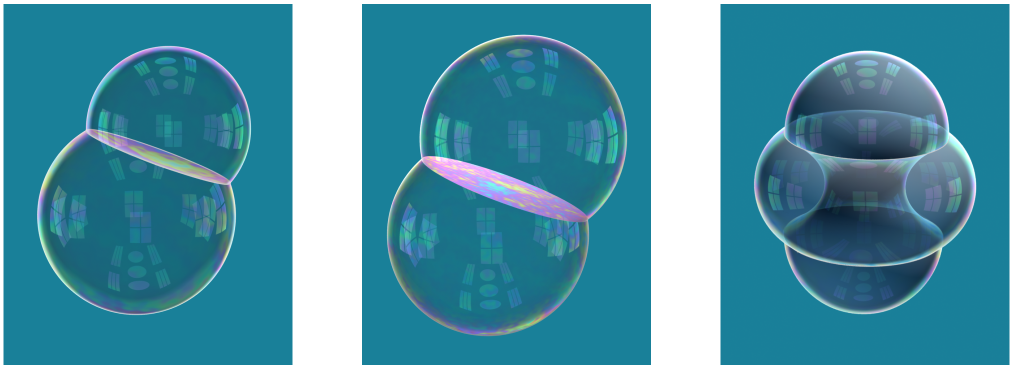

Figure 2 : *The standard double bubble (left) is composed of three spherical caps meeting at* $120°$ *angles along a circle. When the volumes are equal (center) the central film is flat. Proving that this is the optimal double bubble required ruling out strange configurations where one bubble wraps around the other like a belt (right) or even where each bubble is disconnected.*

For any pair of volumes V_1, V_2 there is a unique such standard double bubble; they are related by Möbius transformations of space (inversions in spheres). As is appropriate in Möbius geometry, we will consider a plane as a special case of a sphere. A *spherical cluster* will mean one where each soap film is part of a sphere (or plane); the *nonspherical clusters* of the title are ones where this fails to be true. (Mathematically, we could also consider clusters in spaces of different dimension. Note that Flatlanders' bubble clusters in the plane are always spherical.) Applying a Möbius transformation to a spherical cluster always gives another equilibrium cluster (see [9]) but this fails for nonspherical clusters.

It seems "obvious" that these standard double bubbles must be the solution to our mathematical formulation of the problem for $k = 2$, since they are the only ones ever seen physically. But this was surprisingly difficult to show rigorously, and was first proved [5] in 2002. Mathematically we can build other equilibrium clusters, like that in Figure 2 (right), which follow all of Plateau's rules. These nonspherical clusters seem to all be unstable equilibria, explaining why they are never seen physically.

For clusters of 3 or 4 bubbles, again there is a standard symmetric configuration of equal bubbles, and Möbius transformations give stable clusters with all possible sets of volumes. Weird nonspherical clusters can again be built mathematically, but these are unstable and I conjecture no nonspherical cluster is stable.

The situation is different for $k = 6$. The nonspherical cluster shown in Figure 3 was simulated numerically in Brakke's `evolver` [2], and second-order analysis shows it to be stable. The four large outer bubbles have volume 10, while the two small interior bubbles have volume 1. Essentially we have taken the standard equal-volume cluster of four bubbles, and blown two small bubbles to decorate its central singularity. A single decoration would be a bubble with the combinatorics and symmetry of a regular tetrahedron, while a double decoration cuts that tetrahedron into two pieces. Most efficient would be to cut one corner off the tetrahedron – the resulting cluster would be spherical and have less area, but would be less symmetric. Our cluster instead slices the tetrahedron symmetrically, halfway between two opposite edges, creating two congruent pieces, each combinatorially a triangular prism. The exploded view in Figure 4 (left), where all the faces of the front bubble have been removed and the inner bubbles have been colored in blue and green, helps to understand this cluster better.

Fred Almgren and I tried a similar experiment, physically blowing two small bubbles at the center of a tetrahedral wire frame; while we could achieve this nonspherical configuration, it was only barely stable, and depending on how we oriented it, gravity would pull one of the bubbles to a corner of the other.

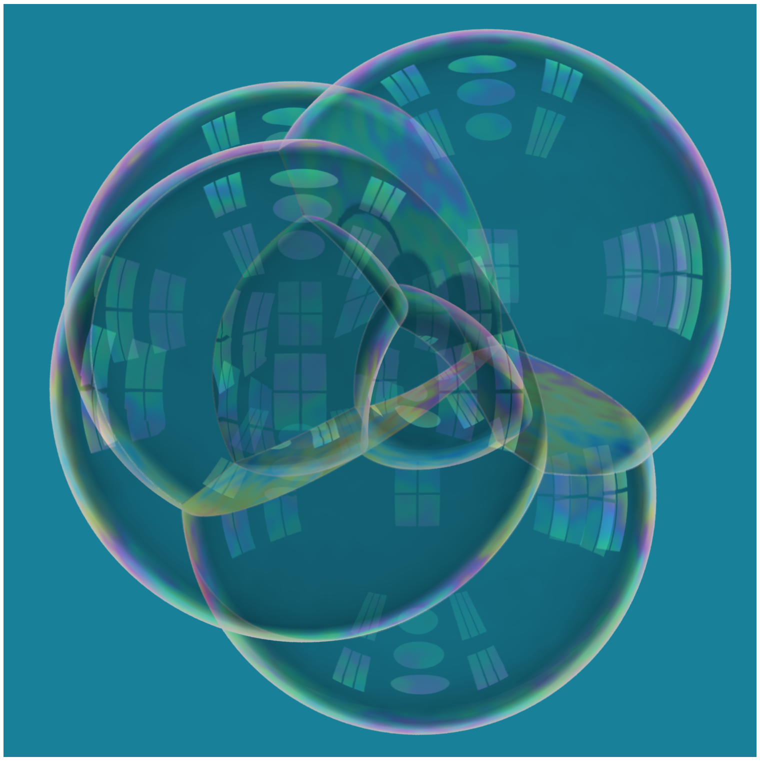

Figure 3: *This is a stable nonspherical cluster of six bubbles. The four outer bubbles have 10 times the volume of the two inner ones. By symmetry, the central film between the inner bubbles is a minimal surface (with mean curvature zero); the picture makes it clear that it is not a flat plane but instead a curved saddle surface.*

My computer graphics rendering (Figure 3), using a custom soap-film shader for Renderman, was produced in 1990 for [1]. It was used for cover of the second edition of Frank Morgan's book [6] and again for the fourth edition. In 2013, it was one of the 24 mathematical images featured in Günter M. Ziegler's book "Mathematik – Das ist doch keine Kunst!" [10], whose title can mean either "Mathematics – There's nothing to it!" or "Mathematics – That can't be art!". This inspired me to write more about it myself.

The symmetry group of the cluster is $2{*}2$ in the Conway–Thurston orbifold notation, with two perpendicular mirror planes and a rotary reflection of order 4; the four outer bubbles are congruent, as are the two internal ones. Congruent bubbles have equal pressure, so films between them are minimal surfaces (with mean curvature zero), perhaps flat planes. Two of the films between the outer bubbles (at the left and right of the figure) are indeed flat – they lie in the mirror planes. The central film between the interior bubbles, on the other hand, is a minimal surface which – as the picture clearly shows – is not flat but instead saddle-shaped. Thus the cluster is nonspherical.

The four large outer films are very close to spheres, but we can prove they are not exactly spherical. For this, note that the Cauchy–Kovalevskaya theorem, a basic result in partial differential equations (PDEs), applies to give the following fact about bubble clusters: If two of the three films meeting along a Plateau border are spherical then so is the third. (The idea is that the two spheres meet at 120° along a circle; there is a third sphere – as seen in an appropriate double bubble – which could be the third CMC surface here, and the theorem says that the Cauchy data for the PDE determines that surface uniquely.)

If one of the outer films in our cluster were spherical, by symmetry all four would be. The six films between them would then be flat planes, so the whole outer part of the cluster would look exactly like the symmetric cluster of four bubbles. But this is not compatible with the interior bubbles: by the tetrahedral

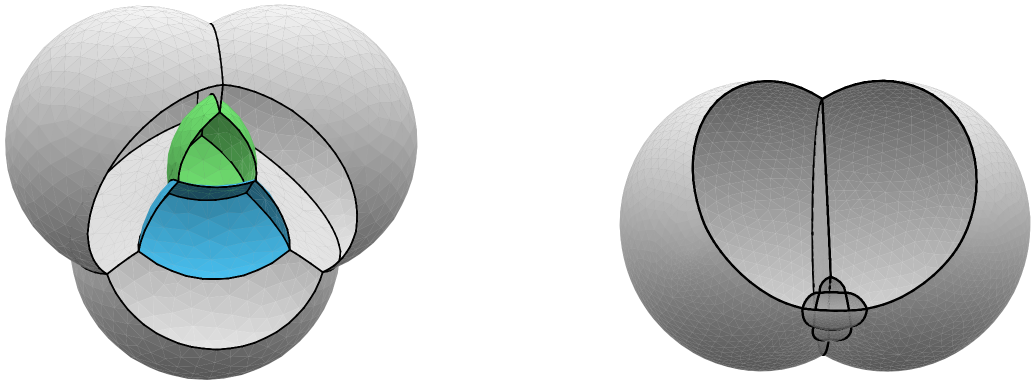

Figure 4 : *In this exploded view (left) of the same cluster, the faces of the front bubble have been removed to better see the internal structure. I suspect there is a similar cluster of just five bubbles (right, again shown with the front bubble omitted) but have not been able to simulate it.*

angle condition, their edges cannot lie in the required planes. (A similar argument shows rigorously that the central film is not a plane; this is of course obvious in the picture.) Further, one can argue that there is no spherical cluster with this combinatorics, even without any assumption of symmetry.

Although in 1990 I conjectured this might be the smallest stable nonspherical cluster, I no longer believe this. Although I have not been able to find a nonspherical cluster of $k = 5$ bubbles, I now believe one must exist. Figure 4 (right) shows an attempt to decorate one of the vertices of a standard triple bubble with a pair of small bubbles. The same combinatorial arguments show no cluster like this can be spherical. By Plateau's rules, any vertex in a cluster looks tetrahedral in a magnifying glass. Thus since the double decoration worked at the central vertex to give our $k = 6$ cluster, a similar decoration should work at any vertex, if done at a small enough scale. Unfortunately, my numerical simulations have not yet found such a cluster with just five bubbles – the one in Figure 4 (right) is not in equilibrium.

References

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