Voronoi Phyllotaxis Tiling on the Fermat Spiral

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Abstract

We developed Voronoi Tiling on the Fermat Spiral using continued fractions and applied this system to architecture and product design. We produced physical prototypes of them using a 3D printer.

Phyllotaxis on the Fermat Spiral

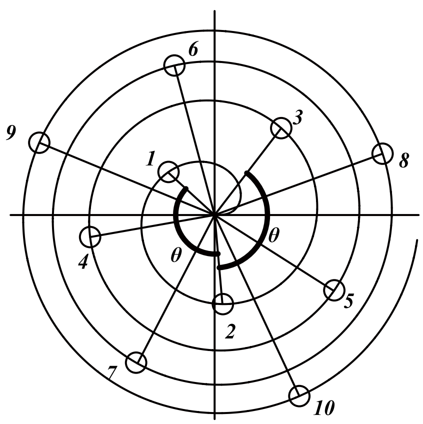

Figure 1: *Plot generatrix on the Fermat Spiral.*

It is well known that the pattern of seeds in a sunflower is based on the Fermat Spiral, as expressed in the polar equation $r^2 = a * \theta$: see [1][2][3]. Figure1 shows the principle for determining the center of seeds as a generatrix. We place numbers on the seeds one by one from the center towards the outside. When θ is $2\pi(2\text{-}\tau)$ (where τ denotes the Golden Ratio), the arrangement of seeds becomes the ideal phyllotaxis.

Our eye can't help tracing many spirals. Every spiral consists of congruent numbers modulo the Fibonacci Number. The principal convergents of the Golden Ratio consist only of the Fibonacci Numbers. The even-order convergent minus τ is always opposite in sign to the odd-order convergent minus τ. That is why people say that the number of clockwise and counter-clockwise spirals is always any two consecutive terms in the Fibonacci Sequence.

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An attempt at Voronoi Tiling based on the generatrix was already published by Sadoc, Rivier, and Charvolin (2012): see [4]. We derived it independently in 2014 and are able to replace the phyllotaxis angle of $2\pi(2\text{-}\tau)$ with any arbitrary angle. We can determine the closest cells using the denominators of convergents calculated by a continued fraction. In the case of the Golden Ratio, it's enough for us to consider just the seven closest cells, but in case of a different real number, we should consider more than seven cells. So it is fair to say that the Golden Phyllotaxis is the simplest system of all real numbers.

Let us take a look at the sample drawings. Figure 2 is a Voronoi Tiling on the Fermat Spiral based on the Golden Ratio. Cell numbers range are from 1 to 999. The tiling consists of pentagons, hexagons and heptagons. There are no quadrilaterals and octagons.

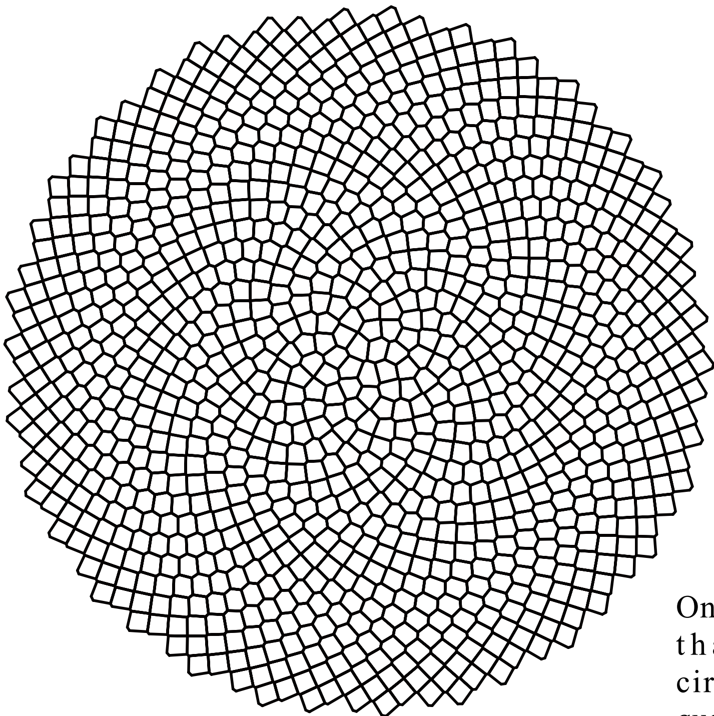


Figure 2: *Voronoi Phyllotaxis Tiling on the Fermat Spiral.*

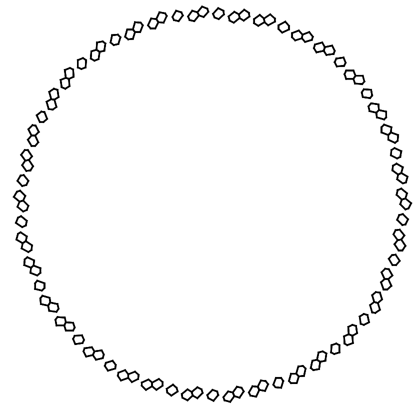


Figure 3: *1D Quasi-Periodic Pattern. A sample: from #1509 to #1597*

One of the author (Hizume) found in 1986 that a phyllotactic division of a circumference made a one-dimensional quasi-periodic pattern, that is, the Fibonacci Lattice: see [5]. Similarly, when we chose suitable series of cells on the same circumference, we achived the same lattice, as shown in Figure 3. We can produce any long sequence from this Voronoi Tiling in such a way.

Figure 5 shows an enlarged view of a portion of tiling. We can identify the numerical order and center of each cell. (After #1,000, the number indicates the lower three digits.) Figure 4 indicates the cell's kind of polygon and cells surrounding it.

For example, you may confirm the configuration of crll #932 in Figures 4 and 5.

873,1017,1072,983,839,
#928 is 5 gon

874,1018,1073,984,840,
#929 is 5 gon

875,1019,1074,985,841,
#930 is 5 gon

876,1020,1075,986,842,
#931 is 5 gon

877,1021,1076,987,843,
#932 is 5 gon

878,1022,1077,988,844,
#933 is 5 gon

879,1023,1078,989,845,
#934 is 5 gon

880,1024,1079,990,846,
#935 is 5 gon

881,1025,1080,991,847,
#936 is 5 gon

Figure 4: *A Data List.*

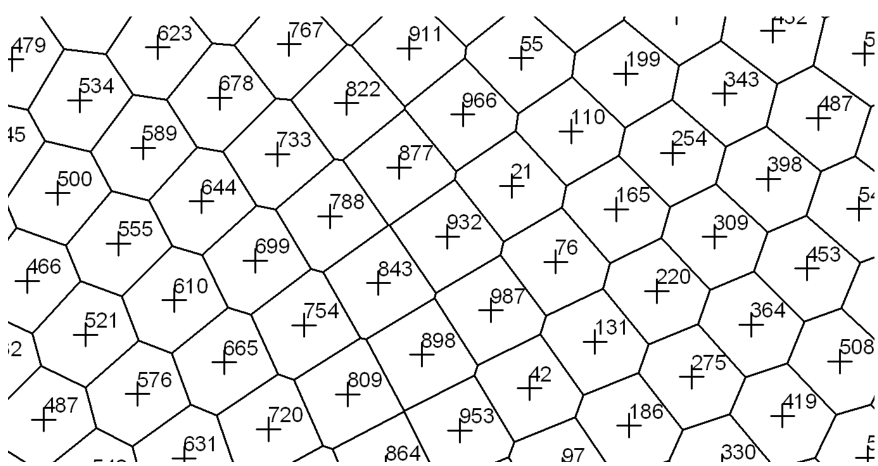


Figure 5: *Enlarged view of a portion .*

We have developed various proposals for applying the Voronoi Phyllotaxis Tiling on the Fermat Spiral as shown in Figures 6 through 9.

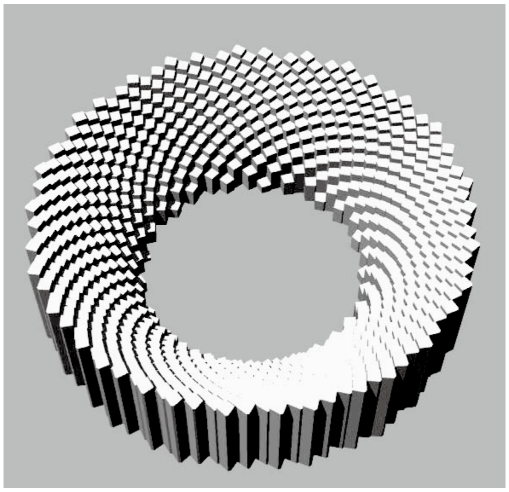


Figure 6: *Fibonacci Colosseum. Everybody can watch the central stage.*

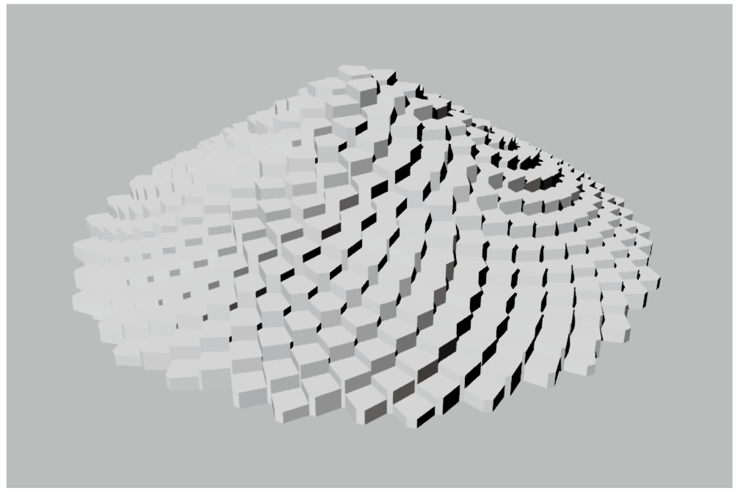


Figure 7: *Mt. Fibonacci for playground or solar generator.*

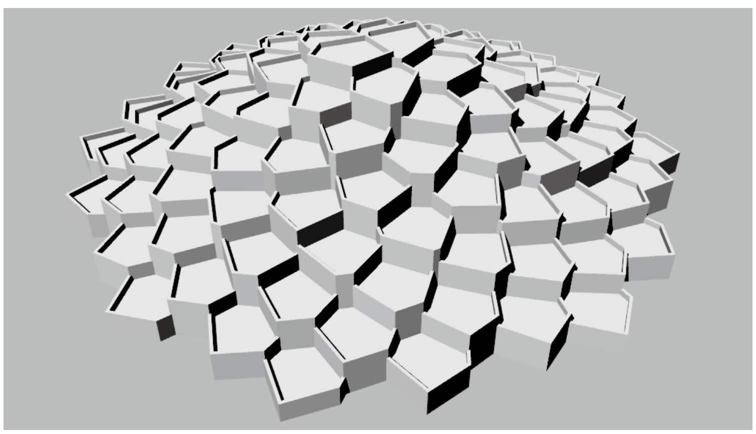


Figure 8: *Fibonacci Fountain or Terraced House.*

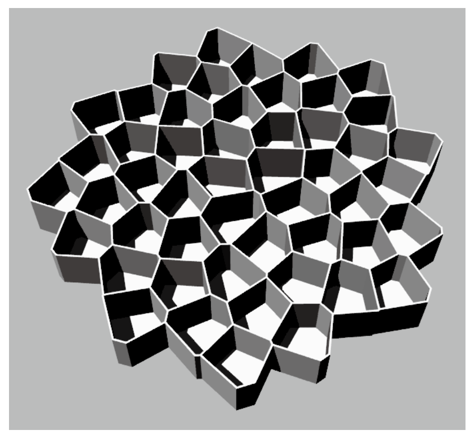


Figure 9: *Fibonacci Honeycomb.*

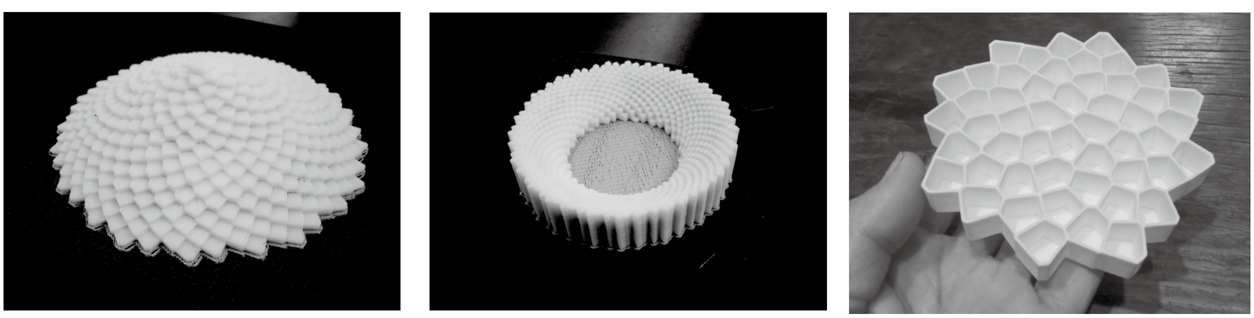
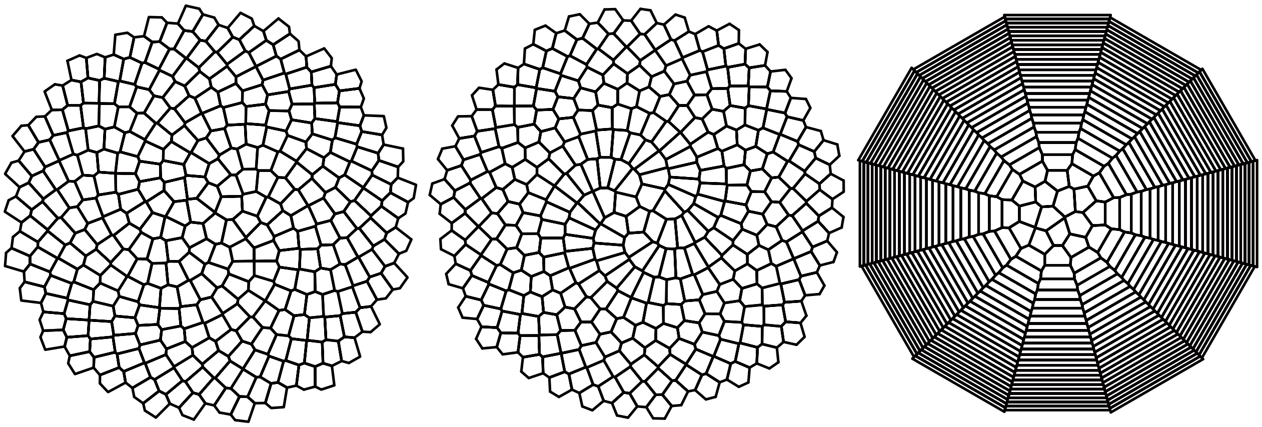


Figure 10: *Prototypes produced by a 3D printer.*

Extension into Other Real Numbers

We extended the above attempts into other real numbers in the same way that the authors extended the "Fibonacci Tornado [6]" to become the "Real Tornado [7][8]". In this endeavor, we made full use of principal and intermediate convergents generated by the continued fraction of the decimal part of a real number. Let us look at some examples.



$\theta = 2\pi(sqrt(2)\text{-}1)$ $\theta = 2\pi(exp(1)\text{-}2)$ $\theta = 2\pi(7/12)$

Figure 11: *Various Voronoi Tiling on Fermat Spiral.*

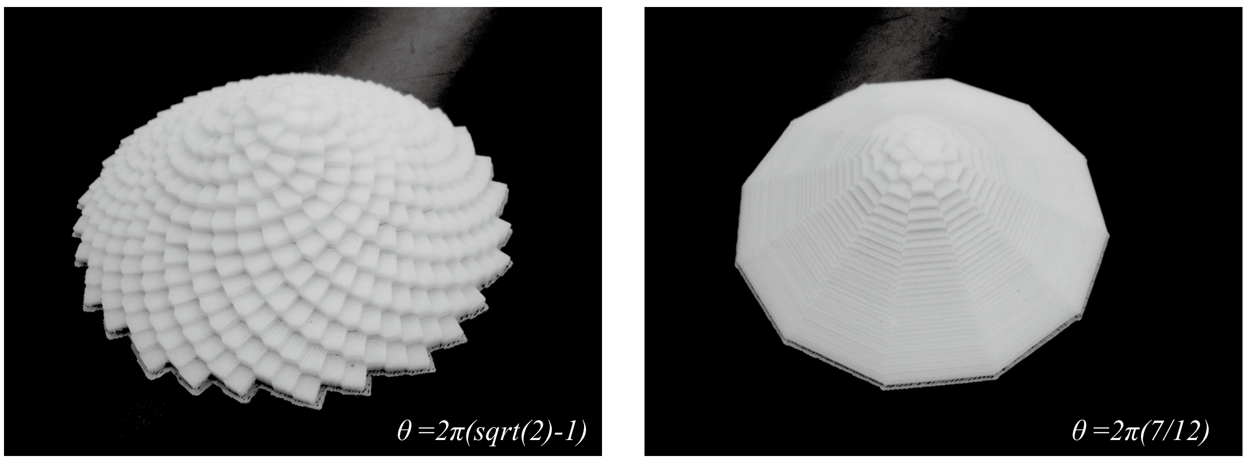


Figure 12: *3D printer models.*

Acknowledgement

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