Dancing Deformations

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Abstract

The performing art of dance employs symmetry in a variety of ways. Often choreographers blur the lines between symmetries or seamlessly morph from one symmetry type to another. This may be seen to be similar to parquet deformations, visual images in which one tiling deforms seamlessly into another. In this paper we look at ways linear or frieze symmetries, common in folk dance and other forms, and planar or wallpaper symmetries used in a variety of dance forms may be switched from one to another in time, with particular attention to the Klein four group. The sequences through danced wallpaper symmetries might make for entertaining mathematical flash mob dances.

Parquet Deformations. The artist M.C. Escher created a number of works in which one tessellation morphs into another. Later in the 1960s the architect William Huff investigated these designs with his students, and received wider attention when explored and written about by Douglas Hoffstadter [4]. Recently Craig S. Kaplan has presented his investigations of them at Bridges [6]. These visual designs usually change seamlessly in a horizontal direction through several tiling patterns. Dance choreographers often utilize similar deformations, in both time and space. In this paper we take a look at linear symmetries in dance, with particular attention to the Klein four group, and begin to examine danced deformations of wallpaper groups.

Dance Symmetry. The great American modern dance pioneer Doris Humphrey famously proclaimed, "Symmetry is lifeless" [5], by which she meant bilateral symmetry within one dancer's body, or else an arrangement of dancers in which the stage left dancers form a mirror reflection of the stage right dancers. These, she said, create balance rather than the tension necessary for engaging compositions. However, there are deep notions underlying symmetry, for example ways in which things are the same or opposite, or ways in which they repeat or change, and the employment of these concepts in choreographic patterns is a much richer subject than suggested by Humphrey's warning to other dance makers. In mathematics, symmetries and the ways they combine are carefully delineated and can provide rich material as well as useful vocabulary for artists. Even Doris Humphrey employed myriad symmetries beyond simple bilateral reflection in one or many bodies in her own choreography. In this paper we will examine certain common uses of symmetry in dance, and aspects of how one choreographic symmetry may morph into another.

The mathematics educator Zoltan Dienes created numerous classroom activities using symmetry groups, both with manipulatives and by having students use their bodies in dance or other kinesthetic activities. Dienes had a lifelong affinity for dance, perhaps due to the influence of his mother, Valeria Dienes, a famous Hungarian dancer who developed a mathematical system for notating dance, and who enrolled her children in Isadora Duncan's brother Raymond's Greek-like communal school, where dance was in integral part of the curriculum [2]. Dienes created many activities using symmetry groups, including groups of order 2,4,6,8,9 and 12, such as in *Mathematics through the senses, games, dance, and art* [1]. Robert Wechsler in his article "Symmetry in Dance," [14] outlined various forms of symmetry in space used by choreographers, and made a plea for dancers to learn the distinctions between kinds of symmetries. When the author and Erik Stern created our first math and dance concert *Two Guys Dancing About Math* in 1990, we included audience interactions on symmetry and a humorous dance with many symmetries, and we began teaching workshops on symmetry and dance focused on the four planar symmetries - translation, reflection, rotation, glide - and their combinations in the Klein Four Group [10].

The Klein Four Group. The Klein four group appears in many guises, often as the symmetry group of the rectangle. For example, corresponding to the four symmetries of the rectangle, there are a total of four ways to insert a credit card in a gas pump's card slot, only one of which works! In *Group Theory in the Bedroom* [3], Brian Hayes delineates the ways a rectangular mattress may be flipped to cycle through all four possible sleepable positions, and compares these to the strategies for rotating tires on a car. In [10] we mention that this group may be modeled with a doll performing ½ turn rotations around three mutually perpendicular axes: ½ spins or pirouettes around the vertical axis, ½ somersaults around the right side/left side axis, and ½ cartwheels around the front back axis, as in Figure 1. These three motions may also be modeled with the hand, since the three major joints in the arm allow easy rotations of the hand (or modeled with the entire body by those of more gymnastic bent!). They constitute three non-identity motions of the doll or hand, with the starting position, or zero motion, representing the identity, and they form a symmetry group of order, or size, four, in which each element is its own inverse. (This self-inverse aspect is key to the Klein four group's equivalent representation as the direct product of two groups of order 2, denoted $Z_2 \times Z_2$). We also say that these motions "act on" or transform the set of four positions shown in Figure 1, transforming one into another.

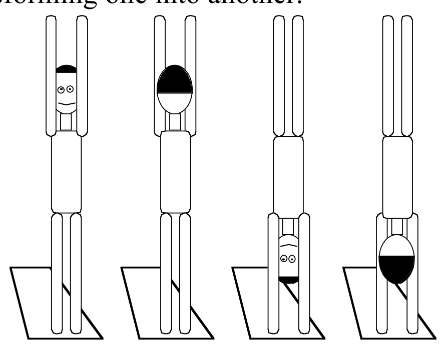


Figure 1: *Half spins (or dives) and the Klein four group*

Competitive diving, a sport with artistic elements similar to gymnastics, also includes aspects of the Klein four group neatly illustrated by these same motions. A diver leaves the diving board either facing front or back, as in the first two positions in Figure 1, and almost always enters the water arms outstretched, head down, and hands first, again facing either front or back, as in the last two positions in Figure 1. However, water entrances that are feet first, facing front and back, are also legal entries, though seldom used in high-level competition. Also, in platform diving it is common for divers to leave the platform from a handstand, either facing forward or back. Thus any of the four positions shown may be starting or ending positions of the body in competitive diving. Of course, divers commonly do much more than execute moves as simple as ½ spins or ½ somersaults in the air; these motions merely represent the total change in position between the beginning and end of the dive.

In [10] the authors present activities for several participants exploring four isometries of linear frieze patterns: translation in the direction of the frieze line (T), 180° rotation (R) around a vertical axis perpendicular to the frieze line, mirror reflection through a plane perpendicular to the frieze line (M), and glide reflection in the direction of the frieze line and through a vertical mirror plane containing the frieze line (G). Here G is also used to represent a reflection with zero translation. In [10] we show how these may be seen to combine to give the Klein four symmetry group, as in the table in Figure 2. In these activities we used symmetries in a line in order to simplify the activity; Christine von Renesse has adapted these activities to have students explore a larger set of planar symmetries, not all in a line [8]. These symmetries are assumed to operate on one dancer's body shape and orientation and give as a result a second dancer's position, all in the direction of a given line; in this way T, G, M, and R actually represent classes of symmetry operations rather than specific motions. We might show these symmetries using the letters p, q, d, b, and E to represent the dancers' bodies as seen from a bird's eye view above, where, for example, the stem of the p represents a dancer's right arm extending to the dancer's right, with the dancer facing to the right side of the page, as in the diagrams in Figure 2. The letter E represents a right-facing dancer in a position of bilateral symmetry, where there is a vertical mirror plane from front to back of the dancer's body. For example, suppose a dancer p facing forward in the direction of the horizontal frieze line, is reflected through a mirror in front of the dancer, resulting in a second dancer's body position q, and then a reflection of the second dancer in a plane parallel to the first reflection plane gives a third dancer's position, also p. These might be shown as follows: p | q | p, as on the top right of Figure 2. The result is a position that could have been obtained by operating on the first dancer's shape with a translation. However, the two mirror reflections are not identical, so the composition table really is showing how one class of symmetry operations combines with another class (in this case, M followed by M). However, if the mirror planes are identical, the result is the first dancer's position, herein also considered to be a translation by distance zero. Figure 2 also shows a reflection and then a rotation resulting in a glide. Linear frieze symmetries are assumed to extend in a line to infinity, both forwards and backwards; choreographers often suggest a similar idea by having lines of dancers appear on one side of the stage only to disappear on the other. "Circular" folk dance formations in which the dancers move in a circle may employ similar symmetries. These might be thought of as approximately regular polygons in which the vertices are the dancers, the formation thus exhibiting either the discrete cyclic or dihedral symmetries possible for such patterns. They might also use linear frieze symmetries that are only approximate; for example, a glide reflection along an edge of the polygon, would not be directly in line with the next edge. In such circular dances the translations of frieze symmetries have essentially been converted to rotations around the circle's center. Each of the sixteen entries in the Figure 2 table might also correspond to one of the sixteen possible ways to begin a dive and then enter the water, where, for example, T represents beginning a dive standing upright and facing forward.

		Translation	Rotation	Mirror	Glide
		T	R	M	G
Translation	T	T	R	M	G
Rotation	R	R	T	G	M
Mirror	M	M	G	T	R
Glide	G	G	M	R	T

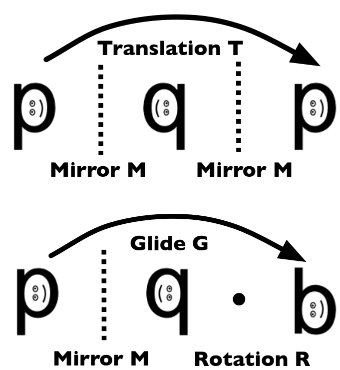


Figure 2: *The Klein-4 composition table*

Switching Symmetries. In [10] the authors note that it is possible for two standing dancers to move smoothly between each of the four symmetric positions and two others, by moving to positions of bilateral symmetry within their own bodies, as shown in Figure 3. Thus, two dancers in linear translation symmetry, symbolized by pp might move to positions of individual bilateral symmetry EE, and then to glide reflection orientation, pb, without intervening positions of asymmetry. Within glide symmetry, both facing the same direction as in pb, one dancer may turn 90° to the right side, the other 90° to the left, and move smoothly to 180° rotational symmetry by passing again through positions of bilateral symmetry within each dancer's body, shown in Figure 3 along the bottom right edge of the diagram. This is possible because dancers facing opposite sides and exhibiting bilateral symmetry in their own bodies are related simultaneously by glide and 180° rotation symmetry. From positions connected by 180° rotation, such as pd, facing each other, two dancers may move through bilateral symmetry to mirror, then both turning 90° to the same side, return to translation, as shown in Figure 3.

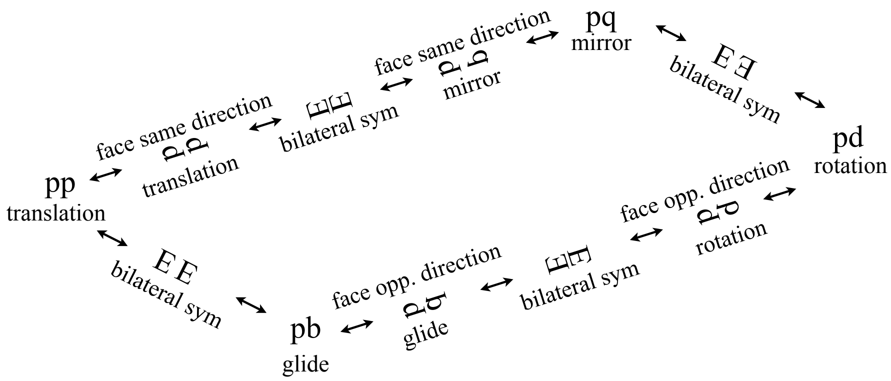


Figure 3: *Linear symmetry switching ("bilateral symmetry" refers to each dancer's individual shape).*

Many dance forms involve long lines or circular formations of dancers. For example, as mentioned above, folk dances performed in a circle use approximate linear frieze symmetries. Thus we could also expand the diagram in Figure 3 and look at how passing through bilateral symmetries in individual dancers allows lines of dancers to seamlessly morph one of the seven linear frieze patterns to another, as shown in Figure 4, which indicates the International Union of Crystallography notation and John Conway's footstep notation for the frieze patterns. See [12] and [13] for accessible explanations of these notations.

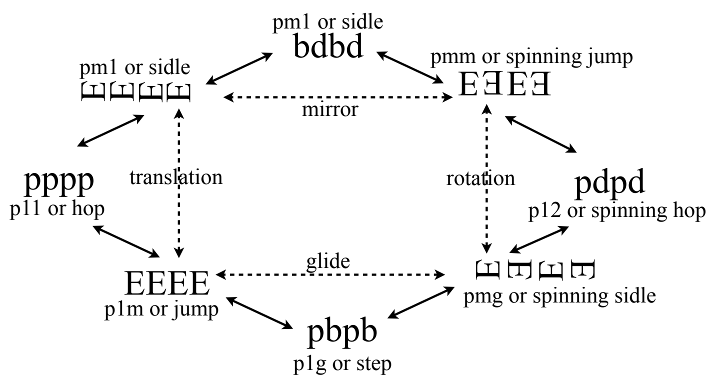
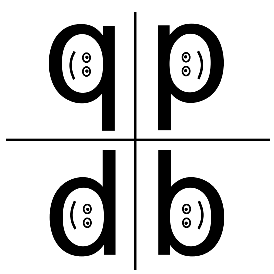


Figure 4: *Switching frieze symmetries*

In this diagram positions of bilateral symmetry within each dancer's body use the letter E, and are the transitions between patterns in which individual dancers do not exhibit bilateral symmetry. Interestingly, it is possible to seamlessly deform through all seven frieze symmetries, returning to the starting position, using these transformations. This might make an entertaining dance sequence for a mathematical flash mob! Additionally, it is possible to move between the four E-type individual bilateral symmetry frieze patterns shown while maintaining translation, mirror, rotation, and glide symmetries, as indicated by the dotted lines. In truth, choreographers and dancers constantly switch symmetries without attempting to maintain symmetry; quick movements are usually too fleeting for most observers to catalogue which symmetries are present or absent, and choreographers are often more interested in getting to new positions or in breaking symmetry for aesthetic reasons. Many folk dances use these patterns in circular formations. For example, the popular French Canadian partner dance La Bastringue moves through sidle (pm1), hop (p11), and spinning hop (p12) formations. The Swedish couple dance Hambo begins with partners facing forward in a circle moving in mirror image to their partners, or jump symmetry (p1m) for the overall circular pattern, after which the couples spin around each other in approximate spinning hop formation, with the man and woman executing identical movements one beat apart: the woman steps R then L on counts 3 and 1, and the man steps R then L on counts 1 and 2 of 3-beat music (the man's steps are thus a translation forward in time of the woman's by 1 beat).

Another aspect of the Klein four group is that the entire group table may be read out from a collection of moving dancers. This activity is best accomplished in the manner described in detail in [10] by dancers moving with one leader improvising in slow motion and the others following according to the appropriate symmetry operation, the dancers also freezing in position to allow observers to see the symmetries at play with more clarity. In the sequence p | q • b, for example, we have, from left to right, a reflection followed by a 180° rotation, resulting in a glide symmetry between the first person, p, and the last, b. Nine of the operations in the Klein four table may be seen here: for example rotation followed by glide is seen in the sequence from q to b to p. Or p followed by q followed by p gives p as a result, showing two parallel mirrors result in a translation. The seven operations in the table involving translation are missing. If instead we have a group of five dancers, for example, move in the sequence glide, mirror, glide, mirror, shown by p b d q p, all sixteen compositions are present, though not easy to see.

We might examine a different formation that demonstrates the Klein four group, as shown on the left in Figure 5. The shape p created by the dancer in quadrant 1 is reflected across the vertical axis to give q in quadrant 2, across the horizontal axis to give b in quadrant 4, and rotated 180° to give d in quadrant 3. The three operations of reflection in the two axes, rotation by 180°, along with the identity operation of no motion at all also illustrate the Klein four group. If the dancers move continually using these symmetries, we can see all sixteen compositions in the group table as the dancers move. For example, reflection of p in the y-axis gives q, and that followed by rotation of 180° gives b, equivalent to a reflection in the x-axis.



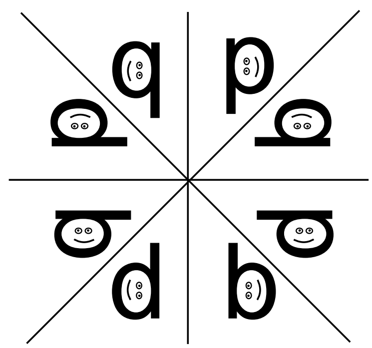


Figure 5: *Four quadrant Klein four group and dihedral group D_4.*

This four-quadrant Klein four dance pattern is useful in education as it is a kinesthetic formation for studying the transformations of reflection in *x*- or *y*-axes or 180° rotation about the origin which replace *x* and/or *y* coordinates by their negatives. It also easily allows exploration of various linear transformations such as rotations other than 180°, dilations or shears, as dancers move in or away from the origin for dilations, or shear in one direction or another – all common actions in group choreography in many dance forms. The diagram on the right shows the symmetry group of the square for eight dancers, the dihedral group D_4, in a formation common in square dance or English Country dance. It also might be educationally valuable, as the reflections across the lines $y = x$ and $y = -x$ model the switching of *x* and *y* coordinates, as in the derivation of inverse functions; again, this might require moving slowly or freezing and examining the positions, rather than hoping to glimpse them as they briefly appear in specific square dances. As before, we might also realize the full D_4 group table from the formation itself. However, we should not forget that dance provides us the unique opportunity to see these symmetry operations in action, rather than as static illustrations, as well as to kinesthetically experience them.

For dancers in circle formations, common in folk dances, the only possible symmetry groups are the dihedral or cyclic groups. This fact is sometimes known as Leonardo's Theorem, for Leonardo da Vinci, who catalogued all the ways to maintain the symmetry of a central building [7]. Figure 6 shows a sequence of positions for four dancers in a circle or square, cycling through each of the non-trivial (that is, not asymmetrical) positions. Similar designs and sequences work for larger numbers of dancers in circular (or polygonal) folk dance formations.

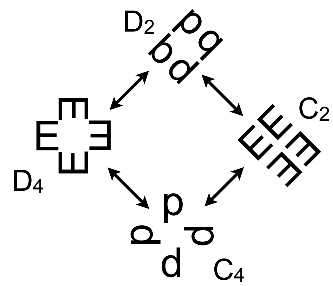


Figure 6: *Four patterns with the possible symmetry groups for four dancers in a square.*

These versions of Klein four, dihedral, or cyclic group formations in two dimensions might lead us in the direction of looking at deformations of larger dance designs, exploring the use of these symmetry switches for large groups of wallpaper patterns of dancers or dance tessellations. Figure 7 shows sixteen dancers following a leader in shape p, each of the followers either reflecting or rotating that shape 180° (each dancer imagining that the leader is directly to the front). With only these rules there are thus 2^{16}, or over 65,000 possible configurations for these sixteen dancers, mostly asymmetrical. If a leader is in position p, then followers in form p, q, d, and b represent translations, reflections, rotations, or glides of that leader's shape respectively, again assuming the leader is directly in front or behind the follower; several tessellation patterns based on these were explored in a workshop in Bridges 2011 [11].

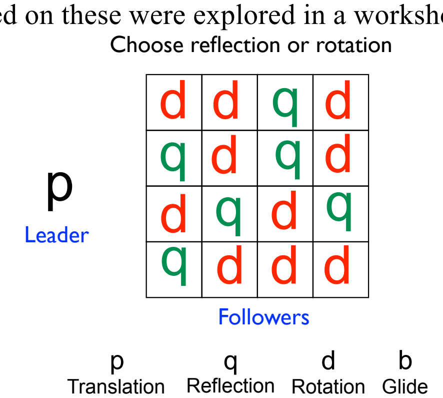


Figure 7: *Sixteen dancers following a leader*

Wallpaper Deformations. Figure 8 illustrates the twelve wallpaper patterns that use zero, 90°, or 180° rotations for sixteen dancers, similar to the sequences using the four planar symmetries in a line and the seven frieze groups. As before E represents a body shape with individual bilateral reflection symmetry. The octagon on the left, with designs joined by solid arrows, was formed by duplicating horizontal rows of the octagon in Figure 4. The sequence through these eight figures uses six of these twelve rectangular wallpaper patterns, with some repeats of pattern pm. The transitions along the octagon's arrows are the same as for the frieze patterns in Figure 4. We can add in the other six designs, again excluding those with 3-fold or 6-fold symmetries, using the dotted arrows. Thus we can replace the solid arrow sequence pg-pmg-p2-pmm-pm in the octagon by pg-cm-pgg-pmg-p4g-p4-p4m-pmm-p2-cmm-cm-cm-pm. The pm to cm transition involves translations of alternating rows in opposite directions, similarly for the p2 to cmm and pg to cm transitions. The cm to pgg transition uses 90° rotations by dancers in each row, the first and third row p's rotating counterclockwise, the first and third row b's rotating counterclockwise, all the while maintaining glide symmetry within the row; similar alternating 90° rotations are used in the second and fourth rows. The pmg to p4g transition involves similar 45° rotations within each row, all maintaining glide symmetry within the rows. The p4g to p4 transition uses four of sets of four dancers rotating 45° around the set's common center. To get from p4 to p4m, each set of four dancers turn inwards towards their common center, while rotating 45° around the center. And the p4m to pmm transition has each row of dancers alternately turn 45° to face their "partners." Many of these transformations are used frequently in dances for large numbers of performers, though each symmetry pattern might last only seconds; one of the referees noted that, for example, the sequence p4m, pmm, p2, pmg, and pgg may be seen in an online video of "Halloween Progressive Square Dance" [9].

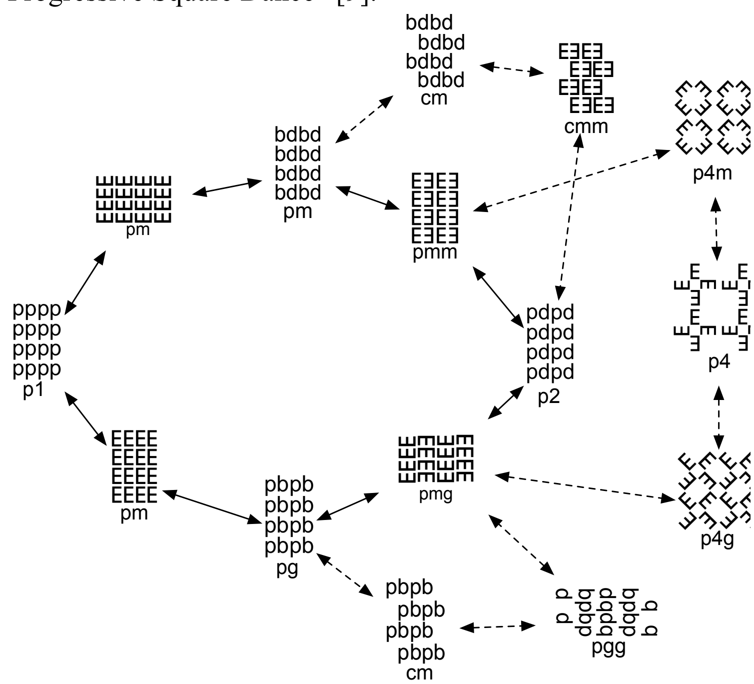


Figure 8: *Some deformations between wallpaper groups of dancers with zero, 2-fold, and 4-fold rotations*

Figure 9 shows the remaining five wallpaper patterns, involving 3-fold and 6-fold rotations, for 24 dancers. The switches between patterns are fairly straightforward. Other transitions are possible between these wallpaper patterns, and as mentioned, actual dances usually use switches among symmetry types by allowing dancers to move through asymmetrical positions, sometimes happening so quickly that they are difficult to follow, sometimes with extended asymmetrical sequences, all depending on the overall intent of the choreography. Since 24 and 16 are both divisible by 4, there are undoubtedly sequences that would include the two sets of twelve and five wallpaper designs shown in an overall sequence of the 17 wallpaper groups, but we will leave that to some future endeavor with actual dancers!

The use of symmetry and asymmetry in dance is actually quite complex. Choreography also often employs non-planar 3-dimensional symmetries. The legs and arms are somewhat similar in length and may form approximate reflections or helical rotations of each other. Cartwheels are sometimes executed with such speed in gymnastic movements that the arms and legs seem to blur into four similar extensions of the body. In the same way that composers use canon, translations in time of a musical phrase or sequence, dancers similarly may perform similar movements sequenced in time in complex ways, as detailed by Wechsler in [14]. Viewed from the side, the body may seem to have another plane of approximate reflection symmetry, and in silhouette it may not be apparent whether a performer is being viewed from the front or the back.

It would be interesting to catalogue not only the symmetry types most commonly employed by various choreographers or dance forms, but the types of transitions between those symmetry patterns. For example, Flamenco duets often utilize C_2 or 180° rotational symmetry between two dancers, suggesting intimate conflict. Square dance and English Country Dance often use symmetries not only from D_4 or C_4, but also a variety of planar symmetries, including dances for 3 and 6 couples using some of the designs in Figure 9. It might be easier to investigate this for dance forms with many constraints, for example many folk dance forms, phalanxes of large choruses in classical ballet, or even patterns used by marching bands, as opposed to improvisatory (or seemingly improvisatory) forms common in some styles of contemporary dance. Again, as mentioned above, these symmetries and the switches between them might make for an entertaining mathematical flash mob.

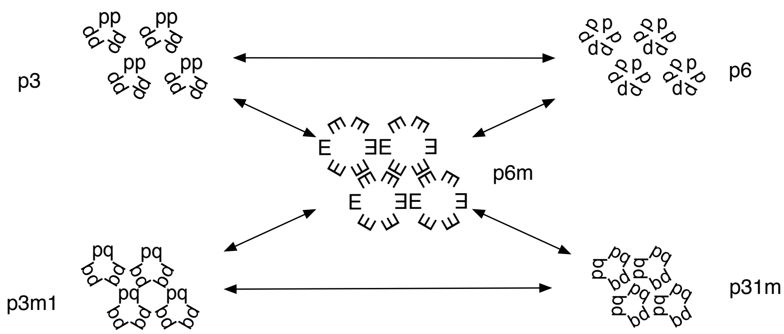


Figure 9: *Some deformations between wallpaper groups of dancers with 3-fold, and 6-fold rotations*

The author wishes to acknowledge the referees' generous and helpful comments and suggestions.

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