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Decorating Regular Polyhedra Using Historical Interlocking Star Polygonal Patterns – A Mathematics and Art Case Study

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Abstract

This paper reports on the application of some historical interlocking patterns for the embellishment of the regular polyhedra (Platonic and Kepler-Poinsot solids). Such patterning can be extended to cover surfaces of some other convex and non-convex solids. In this regard, first the Shamseh n/k star polygon method and the radial grid method will be employed, and step-by-step geometric constructions will be demonstrated, then the girih tile modularity method will be used to explore more patterning designs. Then, the girih tile modularity is used to explore more patterning designs.

1. Introduction

The *regular polyhedra* are highly organized structures that possess the greatest possible symmetry among all polyhedra, which makes them aesthetically pleasing. These solids have connected numerous disciplines including astronomy, philosophy, and art through the centuries. The five that are convex are the *Platonic* solids and the four that are not convex are the *Kepler-Poinsot* solids. They admit the properties that for each (a) all faces are congruent regular polygons (convex or non-convex), and (b) the arrangements of polygons about the vertices are all alike.

Platonic solids were known to humans much earlier than the time of Plato. On carved stones (dated approximately 2000 BCE) that have been discovered in Scotland, some are carved with lines corresponding to the edges of regular polyhedra. Icosahedral dice were used by the ancient Egyptians. There are many small bronze dodecahedra that were discovered from the time of Romans of the second to fourth century that are decorated with spheroids at each vertex and have circular holes in each face. Evidence shows that Pythagoreans knew about the regular solids of cube, tetrahedron, and dodecahedron. A later Greek mathematician, Theatetus (415 - 369 BCE) has been credited for developing a general theory of regular polyhedra and adding the octahedron and icosahedron to solids that were known earlier. The name Platonic solids for regular polyhedra comes from the Greek philosopher Plato (427 - 347 BCE), who associated them with the "elements" and the cosmos in his book *Timaeus*. "Elements," in ancient beliefs, were the four objects that constructed the physical world; these elements are fire, air, earth, and water.

There are four more regular polyhedra that are not convex. Johannes Kepler (1571-1630 CE) discovered two of them, the *small stellated dodecahedron* and the *great stellated dodecahedron*. Later, Lovis Poinsot (1777 – 1859 CE) completed the work by finding the other two non-convex regular polyhedra of the *great icosahedron*, and the *great dodecahedron.* For a comprehensive treatment and for references to the extensive literature on solids one may refer to the online resource *Virtual Polyhedra* – The Encyclopedia of Polyhedra [3].

The Schläfli Symbols (n, m) in the following table present the relationships between the n-gon, as the face of the regular polyhedron, and m, which is the number of faces around a vertex for that polyhedron. By a regular pentagram in this table, we mean a 5/2 star polygon, which is the regular non-convex pentagon. Figure 1 demonstrates the regular solids.

Polyhedron	Schläfli Symbol	Complete Face	Visible Face
Tetrahedron	(3, 3)	Equilateral Triangle	Equilateral Triangle
Octahedron	(3,4)	Equilateral Triangle	Equilateral Triangle
Icosahedron	(3, 5)	Equilateral Triangle	Equilateral Triangle
Hexahedron (Cube)	(4, 3)	Square	Square
Dodecahedron	(5, 3)	Regular Pentagon	Regular Pentagon
Small Stellated Dodecahedron	(5/2, 5)	Regular Pentagram	Golden Triangle
Great Stellated Dodecahedron	(5/2, 3)	Regular Pentagram	Golden Triangle
Great Icosahedron	(3, 5/2)	Equilateral Triangle	Two triangles in Fig 12
Great Dodecahedron	(5, 5/2)	Regular Pentagon	Obtuse Golden Triangle

Table1: *The regular polyhedra*

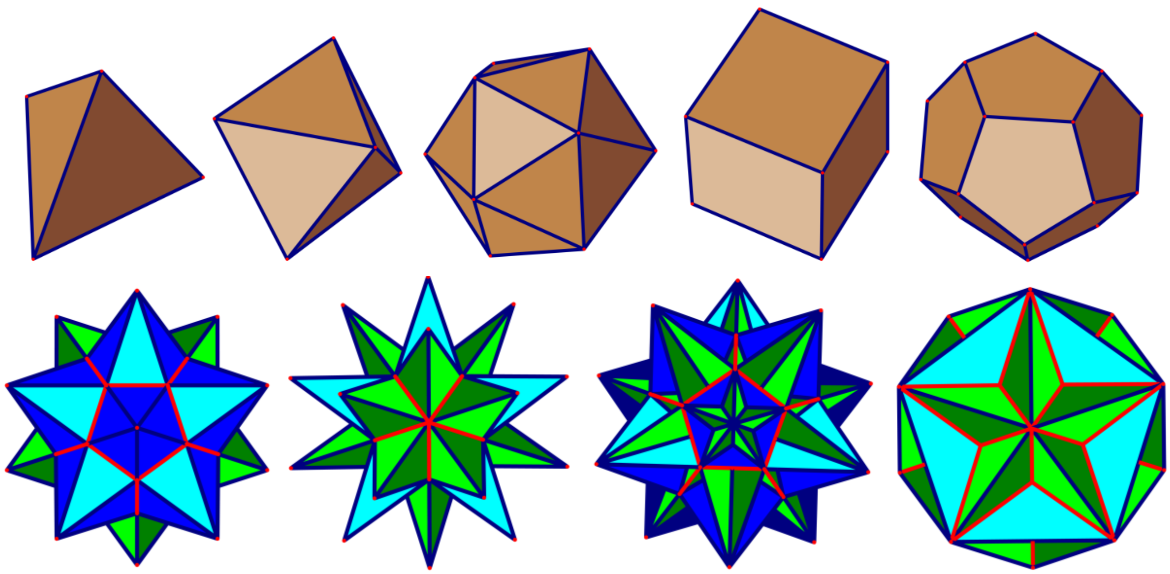


Figure 1: *Top; tetrahedron, octahedron, icosahedron, hexahedron*, and *dodecahedron*. *Bottom; small stellated dodecahedron, great stellated dodecahedron*, *great icosahedron, and great dodecahedron*

The goal of this article is to present methods for the decoration of the regular polyhedra using Persian *girih* patterns. *Girih* (knot in Persian) refers to a (usually) rectangular region consisting of a fundamental region with bilateral or rotational symmetry, for a pattern that includes the nodal points of the web-like geometric grid system and construction lines for generating the pattern.

2. Some Examples in Patterning Solids

There are numerous interesting examples of patterning regular solids. The left image in Figure 2 exhibits the *Screened Icosahedron* created by artist Phil Webster from Pittsfield, Massachusetts. The artwork was presented at the 2013 Bridges Conference Art Exhibition, Enschede, the Netherlands [2]. The right image in Figure 2, the ornamented great dodecahedron created by Richard Kallweit, an artist from Bethany, Connecticut, was presented at the 2014 Joint Mathematics Meeting Art Exhibition, Baltimore, Maryland, USA [2].

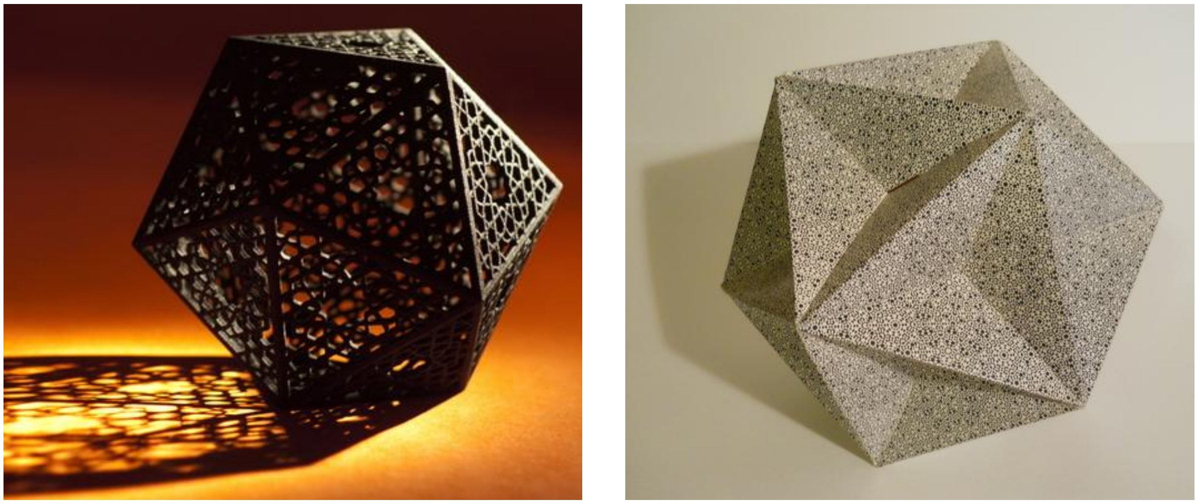


Figure 2: *Screened Icosahedron and embellished Great Dodecahedron*

Captured Worlds by artist Dick Termes (http://termespheres.com) is a set of Platonic Solids that are decorated with fanciful scenes rendered in six-point perspective, which allows an entire three-dimensional surrounding to be projected onto the polyhedra (left in Figure 3). B.G. Thomas and M.A. Hann from the School of Design, University of Leeds, United Kingdom, used the projections from duals to the surface of the Platonic solids, in particular the dodecahedron, in order to decorate the faces of the polyhedra (right in Figure 3) [10].

Figure 3: *Captured Worlds by artist Dick Termes, and an example of the projection of the pattern on the cube to the dodecahedron by B.G. Thomas and M.A. Hann*

During a workshop in the 2010 Bridges Pécs Conference, Hungary, E.B. Meenan of the School of Education and B.G. Thomas of the School of Design, University of Leeds, UK, guided their workshop participants to a process of creating Escher-type tessellations. Then they used presented ideas to extend the workshop into three-dimensions with pull-up Platonic solids constructions that were patterned with Escher's designs [6].

3. Historical Patterns for Embellishment of Solids

It is important to note that since a pattern on one face of an ornamented solid should appear on all faces identically, all the pattern lines should be in complete coordination and harmony with each other in such a way that they can continue from one face to another without any ending or interruption.

There are only a limited number of scrolls (*tumār*) and booklets (*daftar*) from the past that recorded patterns and designs for the decorations of the surfaces of buildings, or as geometric experimentations of interlocking star-polygon patterns. But in general, such designs come with no instruction about the steps of the geometric construction using traditional tools of compass and straightedge or any other tools.

3.1. Patterning Platonic Solids. To decorate regular polyhedra using historical and traditional interlocking star polygonal patterns, one needs to search the documents for ornamented polygons of the equilateral triangle, square, and regular pentagon. Beginning with the dodecahedron and its face, the regular pentagon, the author searched most of the available old documents for a decorated pentagon with the following specifications:

(a) The center of the pentagon coincides with the center of a *k/l* star polygon that covers the central region of the pentagon.
(b) The vertices coincide with the centers of the same or other *k/l* star polygons that cover corners of the pentagon.
(c) Some segments connect star polygons together in a harmonious way to generate a single design.

Mathematically speaking, if all the *k/l* star polygons that are used for ornamenting the pentagon are identical, then since each interior angle of the regular pentagon is $\frac{(5-2)\,\pi}{5}$ = 108°, and since each vertex of the dodecahedron includes three copies of the pentagon, on a successful patterning, a type of star polygon will appear on each vertex that covers 324°. This means the *k/ l* star polygon on the center, which is an *n*-leaved rose that covers 360°, should be constructed in a way that the number of degrees in each leaf divides both 360 and 324, as does their difference, 36. Therefore *k* should be equal to 5*i*, $i \in \mathbb{N}$. Hence, star polygons such as 5/*l*, 10/*l*, 15/*l* and so on will provide proper central designs for the pentagon (that will create concave star polygons of 3/*l*, 6/*l*, 9/*l* and … on the corners of the dodecahedron).

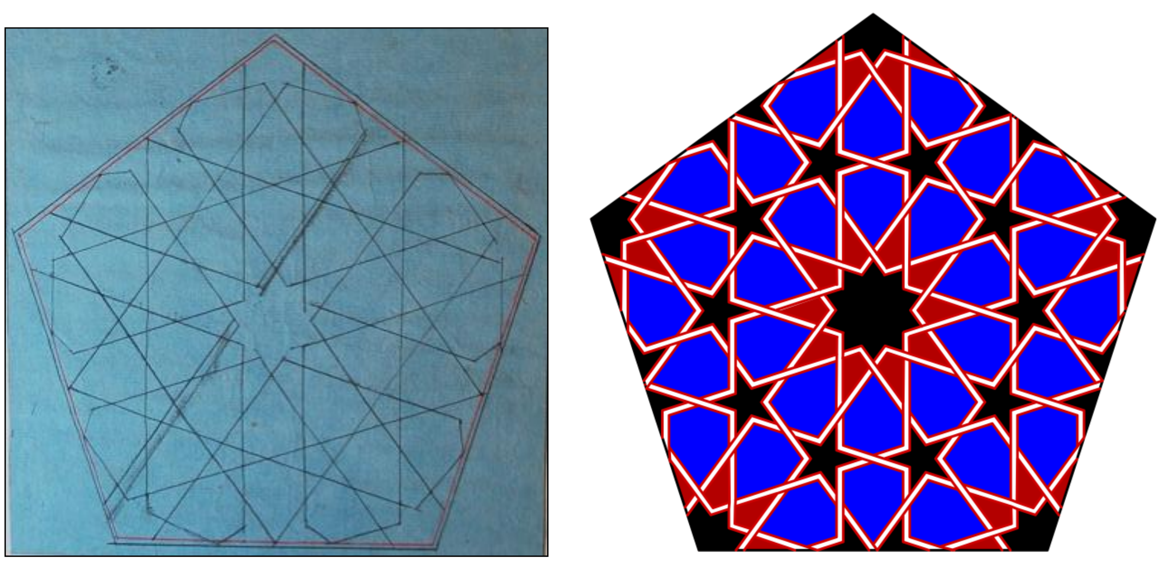


Figure 4: *The decorated pentagon from the Mirza Akbar collection, and the ornamented pentagon created by the author based on the mathematics in the Mirza Akbar ornatemented pentagon*

There are many sources, including buildings of the past, that we can search for Persian traditional patterns and we may find many examples. One source of interest for finding such a pattern was the *Mirza Akbar Collection,* which is housed at the Victoria & Albert Museum, London. This collection consists of two architectural scrolls along with more than fifty designs that are mounted on cardboard. The collection was originally purchased for the South Kensington Museum (the precursor of the Victoria & Albert Museum) by Sir Caspar Purdon Clarke, Director of the Art Museum (Division of the Victoria and Albert Museum) 1896 – 1905 in Tehran, Iran, in 1876. Purdon Clarke purchased them after the death of Mirza Akbar Khan who had been the Persian state architect of this period [7].

The left image in Figure 4 is from the Mirza Akbar collection. As is seen, the constructed lines in this image are not accurate and the pattern looks like a draft. Nevertheless, the design and the 10/3 star polygon at the center satisfy the aforementioned constraints. The image on the right in Figure 4, which was created by the author, using the Geometer's Sketchpad program, illustrates the same pattern but includes interwoven straps, which changes the symmetry group of the pattern from the dihedral group of order 10, D_5 , to the cyclic group of order 5, C_5.

Obviously, for the geometric construction of the pattern, there were no instructions in Mirza Akbar collection, so it was necessary to analyze it mathematically, to discover the construction steps.

One should notice that the radius of the circumscribed circle of the pentagon in Figure 5, *OA*, which is the distance from the center of the pentagon to a vertex, is twice the radius of the circle that is the basis for the 10/3 star polygon at the center (*AM* = *MO* in the top left image in Figure 5). The reason for this is that the two 10/3 star polygons, one at the center *O* and the other at the vertex *A*, are each others reflections under the tangent to the circle at point *M* (see the middle bottom image in Figure 5 that also includes a tangent to the circle at point *N* that is necessary to be used as the reflection line, to complete the star). By following images from the top left to the bottom right a person may complete the design properly.

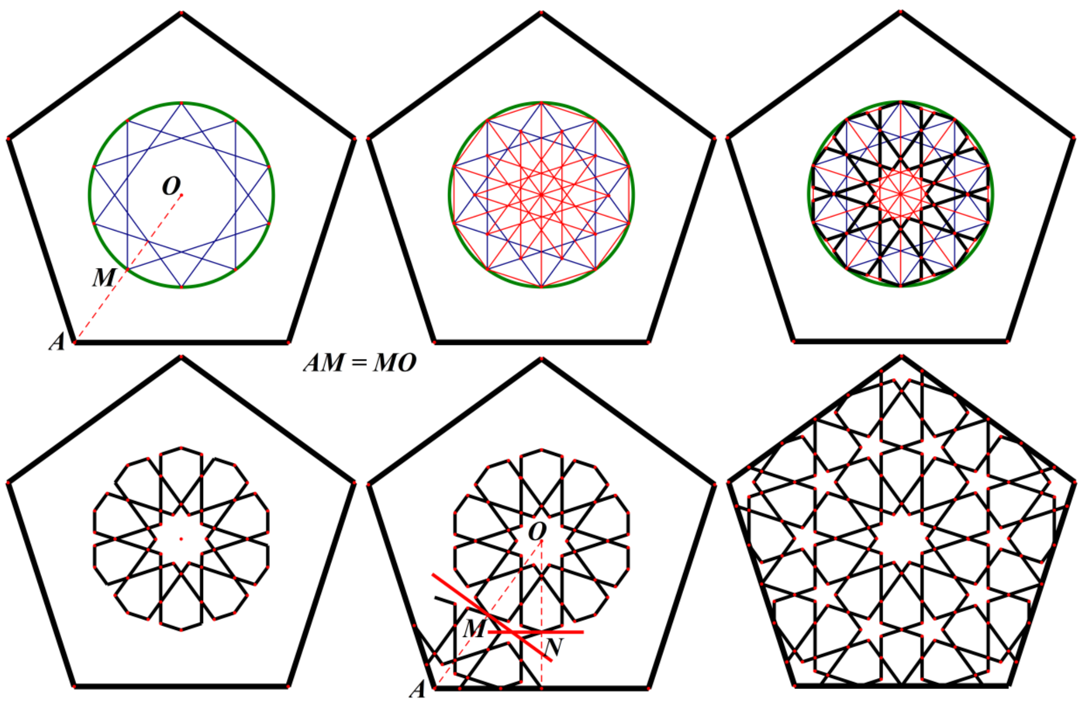


Figure 5: *The steps for the geometric composition of the Mirza Akbar ornamented pentagon*

The photographs in Figure 6 are from a workshop conducted by the author that was presented at the Istanbul Design Center. The workshop was a part of a conference on geometric patterns in Islamic art that was scheduled during 23rd-29th of September 2013 in Istanbul, Turkey. The workshop included the construction of the dodecahedron using the Mirza Akbar ornamented pentagon.

Figure 6: *Photographs from a workshop in the Istanbul Design Center in Turkey*

The next selected polygon for embellishment was the square. It was not difficult to find a decorated square in the Mirza Akbar collection. However, the design, as can be noted in the left image of Figure 7, was a very rough draft with no accuracy on any part of the design, showing only the type of polygons that constituted the structure, but nothing to assist a designer to determine the steps of the geometric constructions. Searching a book by J. Bourgoin [1], plate 118 in this book exhibits the same structure, but the proportions are slighly different from sketch in the Mirza Akbar collections. This book consists of 190 geometric construction plates that appeared in the French edition, *Les Eléments de l'art: le trait des entrelacs*, *Firmin-Didot et C^{ie}*, Paris, 1879. The book does not provide step-by-step instructions for the geometric constructions. Nevertheless, there are underlying circles and segments using thin dashed lines that are instrumental for forming such instructions.

To form the instructions for the pattern illustrated in Figure 8, plate 118 was used but a few steps were changed, to be more in tune with the traditional approaches to complete the ornamented square in Figure 7 (right image).

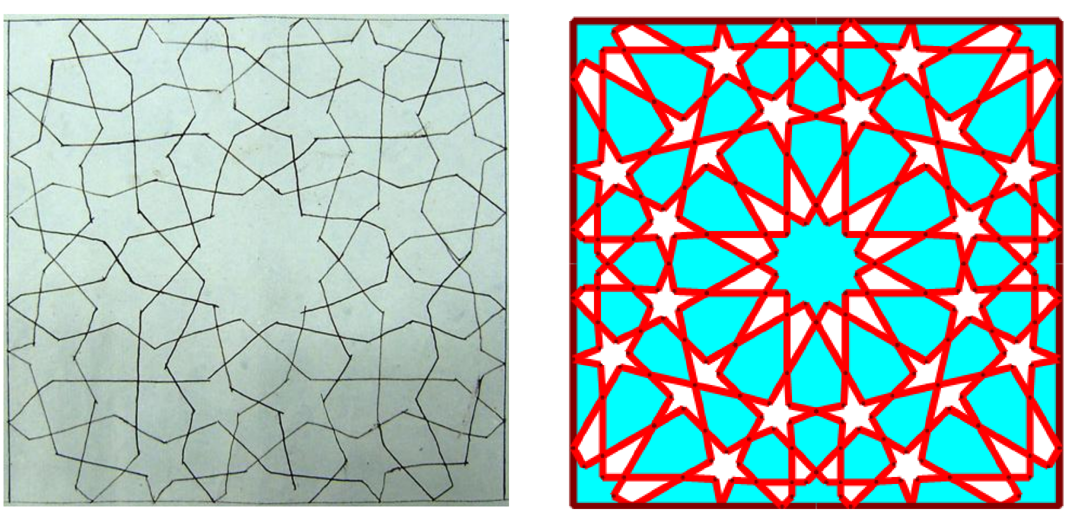


Figure 7: *The decorated square from the Mirza Akbar collection, and the ornamented pentagon created by the author based on the types of tiles that constituted the Mirza Akbar square*

The construction approach, the *radial grid method*, follows the steps that the mosaic designer Maheroannaqsh suggested for another pattern in his book [5].

Divide the right angle $\angle A$ into six congruent angles by creating five rays that emanate from A. Choose an arbitrary point C on the third ray, counter-clockwise, and drop perpendiculars from C to the sides of angle $\angle A$. This results in the square $ABCD$, along with the five segments inside this rectangle, each with one endpoint at A, whose other endpoints are the intersections of the five rays with the two sides of BC and CD of square $ABCD$. Consider C and the dashed segments as the 180° rotational symmetry of A and the five radial segments under center O. Make a quarter of a circle with center at A and radius equal to 1/3 of AC. We repeat all these and the future construction steps for C. Two quarter circles can be constructed at B and D with radius congruent to the distance from B to the intersection of the first ray and BC. The two right angles at B and D are divided each into four congruent angles. The intersections of some new rays emanating from B and D and the previous constructed rays emanating from A and C, as are illustrated in the second top image, are the centers of the circles that are tangent to the sides of the square $ABCD$ and some rays. Dropping a perpendicular from the intersection of the quarter circle with center A and the fifth ray emanating from A to AB will result in finding some new points on other rays for the construction of a quarter of a 12-leaved star with center at A (the top right image). With a similar approach one can construct one quarter of an 8-leaved star with center at B, as is illustrated in the first bottom image. As is illustrated in the bottom middle image, some segments are constructed that connect the four stars in the four corners. Morover, some other segments connect the intersections of the four small circles with the sides of $ABCD$. The last part of the construction, as is seen in the bottom right imgae, is to complete the pattern. The right image in Figure 6 is the result of the reflection of this decorated square under the two sides of AB and AD.

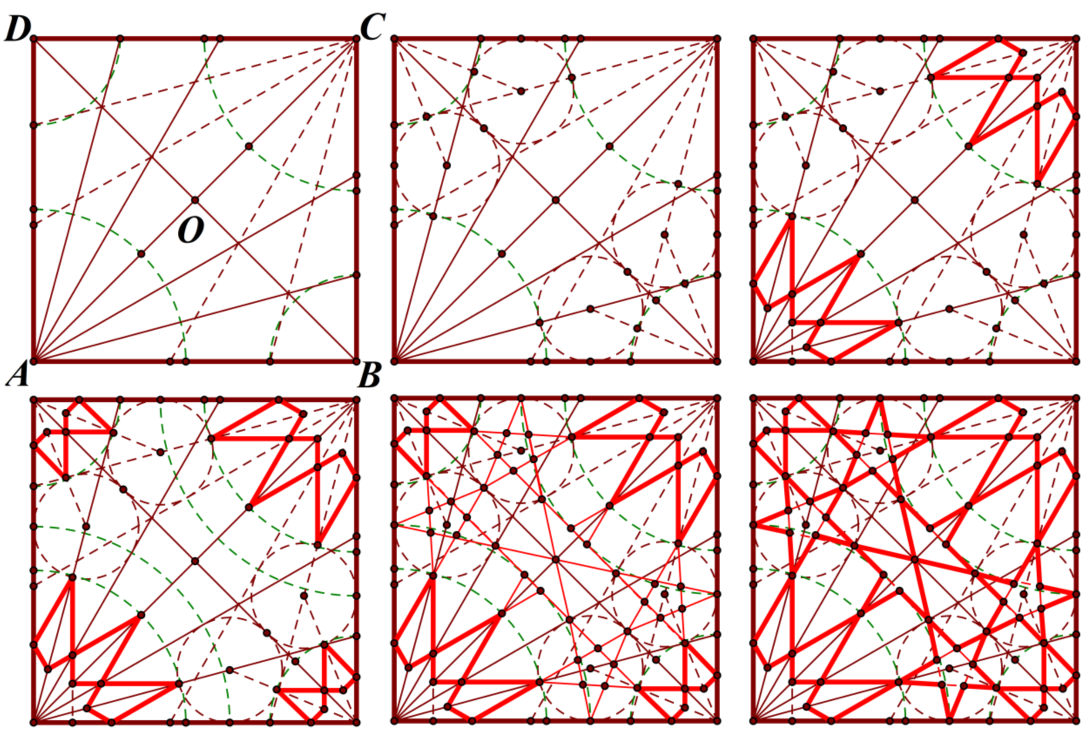


Figure 8: *The steps for the geometric construction of the Mirza Akbar ornamented square*

For patterning the tetrahedron, octahedron, and icosahedron, one needs an ornamented equilateral triangle. To follow the types of patterning in the pentagon and the square, a 12/4 star polygon was inscribed inside the given equailateral triangle. Then similar to the steps in Figure 5, steps were taken to ornament the triangle (Figure 9).

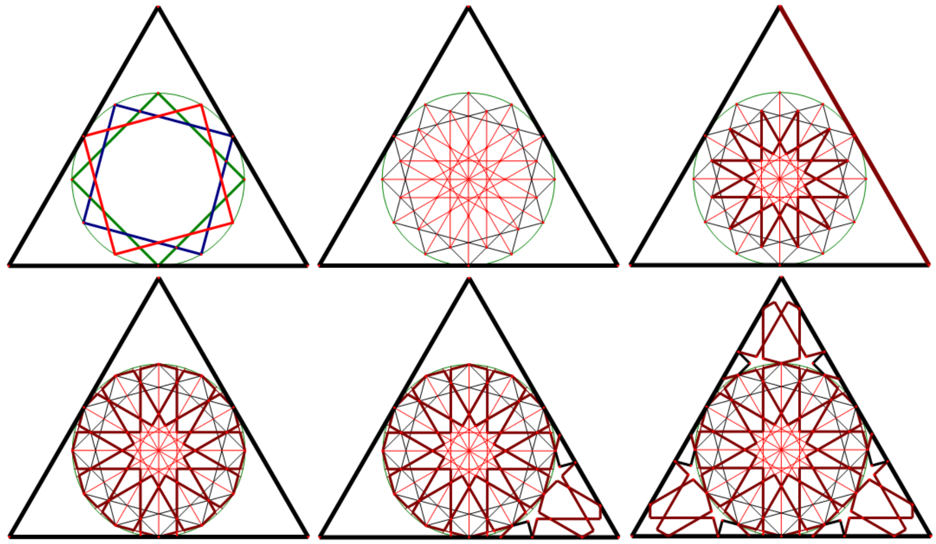


Figure 9: *The steps for the geometric construction of the ornamented triangle*

3.2. Patterning Kepler-Poinsot Solids. The faces of these solids are the regular pentagram 5/2 in Figure 10, regular pentagon, and equilateral triangle. However, faces cross each other and therefore, the physical models have visible faces that are different from the actual faces. The visible faces of the physical models of (5/2, 5), (5/2, 3), and (5, 5/2) are either the *golden triangle* (Δ*ABD* in Figure 10, an isosceles triangle with angles 72, 72, and 36 degrees) or an *obtuse golden triangle* (*ΔBDC* or *ΔADE* in Figure 10, an isosceles triangle with angles 36, 36, and 108 degrees). Therefore, for patterning the above three Kepler-Poinsot polyhedra, we need to ornament these two triangles.

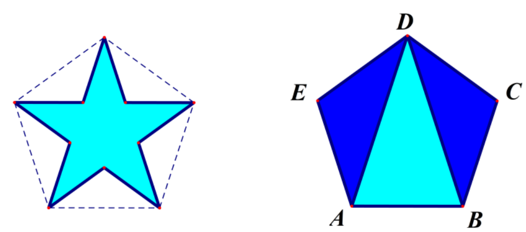


Figure 10:*;The regular pentagram 5/2, and the pentagon divided into the golden triangle and obtuse golden triangle*

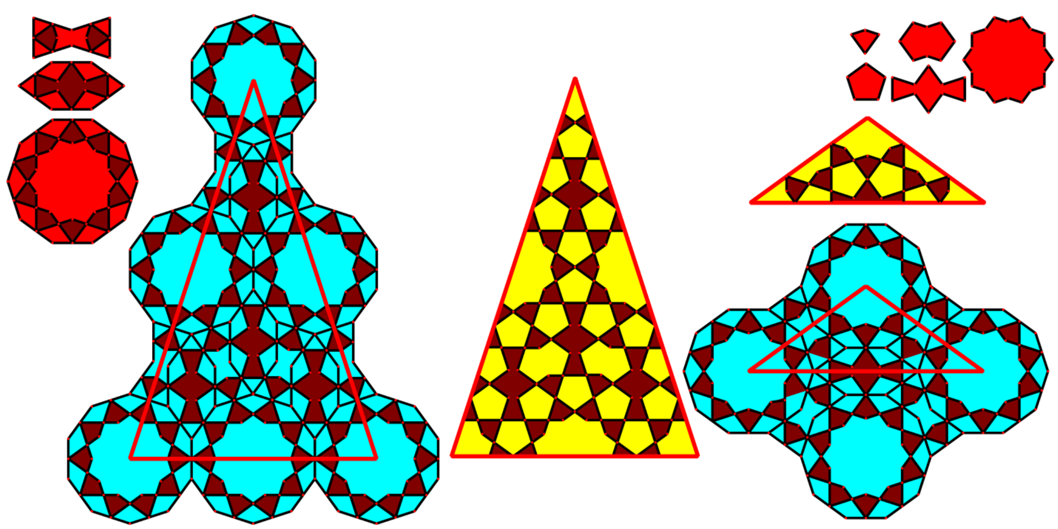


Figure 11: *Ornamenting the golden triangle and the obtuse golden triangle using girih tiles*

For this, the girih tile modularity method presented in [4] was used. In [4] the authors proposed the possibility of the use of a set of tiles, called girih tiles (top left corner of Figure 11) by the medieval craftsmen, for the preliminary composition of the underlying pattern. The pattern then would be covered by the glazed *sâzeh* tiles (top right corner of Figure 11) in the last stage. On the top right image in Figure 11, we see the three girih tiles, which are used to compose the underlying pattern on the two triangles of the golden triangle and the obtuse golden triangle. After finding the pattern, all line segments that constitute the girih tiles are discarded (see the two triangles in Figure 11). Then the *sâzeh* tiles that are presented on the top right corner are used to cover the surface area. For a comprehensive explanation of this and other modularity methods the interested reader is referred to [8].

Figure 12: *Dance of Stars I, the hexagon, bowtie, and the decagram, and the golden triangle*

Dance of Stars I in Figure 12 is one of the four Kepler-Poinsot solids, the *small stellated dodecahedron* that the author sculpted that has been ornamented by the *sâzeh* module tiles. The girih tiles were used to create an artistic tessellation for adorning the surface area of the golden triangle. Similar to panel 28 of the Topkapi scroll in Figure 13, the dashed outlines of the girih tiles were left untouched in the final tessellation. The author also included off-white rectilinear patterns that appear as additional small-brick pattern in the 12th century decagonal Gunbad-i Kabud tomb tower in Maragha, Iran, as is shown in Figure 13. For the hexagon and bowtie girih tiles (middle column of images in Figure 12), these additional patterns posses internal two-fold rotational symmetry. But then this symmetry was followed to create a ten-fold rotational symmetry, in order to cover the surface area of the decagonal tiles as well. It is important to mention that the final tessellation had to conform to three essential rules: (1) Each vertex of the triangles had to be the center of the main motif of the tiling, the decagram; (2) The tessellation should be bilaterally symmetric, (3) The sides of the triangles should be the reflection lines of the motifs located on the edges. Without a thorough mathematical analysis of the pattern, it would be extremely difficult, if not impossible, to create a satisfactory artistic solution.

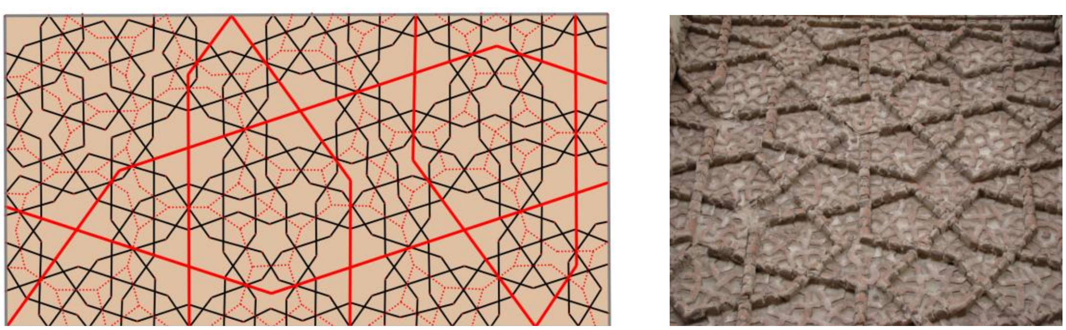


Figure 13: *A rendering of plate 28 in the Topkapi Scroll, and the design on the Gunbad-i Kabud tomb*

Similar to the previous star in Figure 12, *Dance of Stars II* and *III* in Figure 14 are the other two Kepler-Poinsot solids, the *great stellated dodecahedron*, and *great dodecahedron*, which have been decorated by the *sâzeh* module tiles.

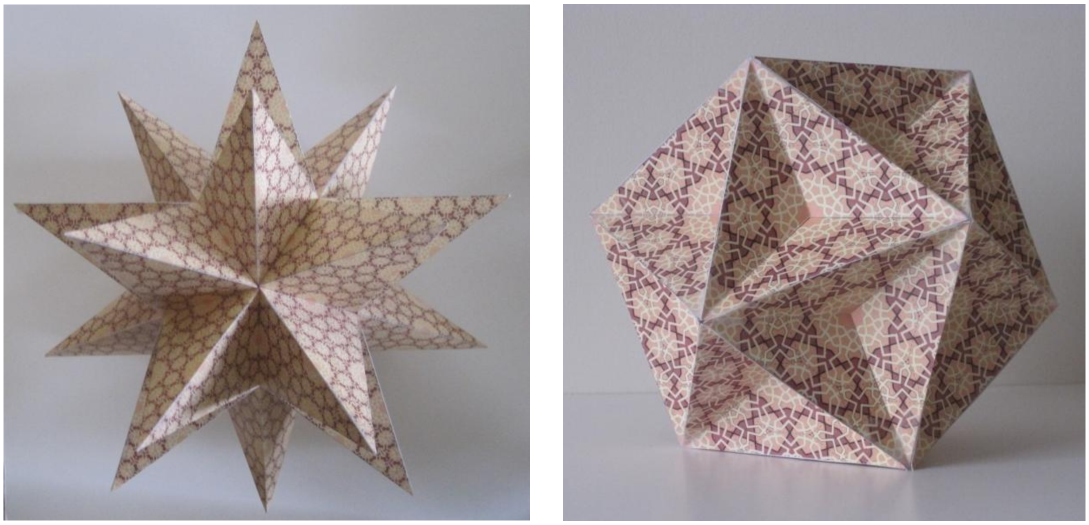
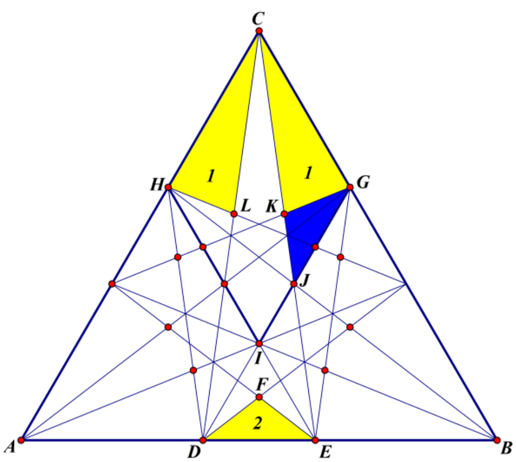


Figure 14: *Dance of Stars II-III, ornamented Great Stellated Dodecahedron* and *Great Dodecahedron*



The triangles that constitute the *great icosahedron* are different from the previous triangles, which were parts of the regular pentagon. So it is possible that we cannot adorn their surfaces using the *sâzeh* module tiles. To construct the two triangles, we need to start with an equilateral triangle and divide the edges at the golden ratio points to create six new vertices on the edges. We then connect them to make "the stellation pattern of the icosahedron" as illustrated in Figure 15 [11]. The two triangles 1 and 2 are the desired triangles (or 1 and Δ *JGK*, which is congruent to 2). It remains an open question whether we can have a sâzeh module tiling solution or if such a solution is impossible.

Figure 15: *The stellation pattern of the icosahedron*

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