Symmetry Groups of Islamic Patterns at the Sultan Qaboos Grand Mosque

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Abstract

In this paper, we analyze Islamic patterns found at the Sultan Qaboos Grand Mosque in Muscat, Oman. We use group theory to analyze these patterns. In particular, we consider dihedral, frieze, and wallpaper groups as possible symmetry groups of the patterns.

The Sultan Qaboos Grand Mosque

Islamic art has been studied and analyzed in many books and papers [1, 2, 4, 8], and even appears as examples in books such as [5] and [6]. In particular, Chorbachi [4] shows how we can use group theory to analyze Islamic art. The Sultanate of Oman has its share of Islamic art very similar to the rest of the Islamic community. However, Oman is the only country where majority of its population adheres to the Ibadi denomination of Islam [8]. Furthermore, Omani history, for the most part, is the history of the Ibadhite movement [7]. Thus, it is worthwhile to analyze many aspects of Islam, including Islamic art, in Oman separately.

The Ibadi school of jurisprudence is known to be purist, resulting in old mosques devoid of embellishments [8]. Oman has many historic monuments and buildings, both functioning and non-functioning, and there is still a need to manage and preserve these historical sites more efficiently [3]. Moreover, the country is opening up to the rest of the Islamic community, relaxing its stance on Islamic art [8].

The grand mosque is a good place to start studying modern day Islamic art in Oman. A documentary film [8] summarizes the importance of the grand mosque as follows: *The newly built mosque gives reference to building traditions from the entire Muslim world. It is a reflection of the fact that Muslims of various factions live together peacefully in Oman today. This architectural principle of eclecticism is concentrated in a single building complex. The architecture of the mosque is understated yet rich in detail. It succeeds in harmoniously presenting the most varied Islamic artistic traditions, coming from different regions and eras*.

The mosque opened in 2001 with a capacity of 20,000 worshippers, as seen in Figure 1. The interior is paneled with off-white and dark grey marble clothed in cut tile work. Arches are adorned with floral patterns with mural panels set in marble. Patterns in niches come from a variety of classical Persian, predominantly Safavid, designs. The ceilings are inspired by Omani forts while the dome has ornate stained glass triangles with a huge Swarovski crystal chandelier with gold plated metalwork [10]. Today, it is the biggest mosque in Oman.

In this paper, we look at patterns that adorn the walls, ceilings, floors, arches, and niches of the grand mosque. We analyze these patterns using group theory and classify them according to their symmetry groups.

Figure 1: *The Sultan Qaboos Grand Mosque.*

Some Group Theory

Abstract Algebra books such as [6, 9] discuss cyclic, dihedral, frieze, and wallpaper groups in detail. In this paper, we use these groups as symmetry groups of Islamic patterns found in the Sultan Qaboos Grand Mosque. Let F be a pattern in the Euclidean plane E^2. The symmetry group of F in E^2 is the set of all isometries in E^2 that carry F onto itself [6]. For completeness, we briefly discuss cyclic, dihedral, frieze, and wallpaper groups here.

D_n, or the dihedral group of order $2n$, is the group of isometries that fix a regular n-gon. The isometries include an n-fold rotation and n reflection axes. The cyclic group Z_n is an index 2 subgroup of D_n. The isometries in Z_n include the n-fold rotation but none of the reflections from D_n. Z_n and D_n are the only finite plane symmetry groups [6].

In transformation geometry, two additional isometries may be considered in order to form an infinite symmetry group of the Euclidean plane. Aside from rotations and reflections, we may also include translations and glide reflections. If we consider an infinite symmetry group whose subgroup of translations is isomorphic to Z, this gives rise to the discrete frieze groups. There are exactly seven frieze groups [6]. In this paper, we use their traditional (Hermann) notation: *p111*, *pm11*, *p1m1*, *p1a1*, *p112*, *pma2*, *pmm2*.

The wallpaper groups, or plane crystallographic groups, are infinite symmetry groups whose subgroup of translations is isomorphic to $Z \oplus Z$. There are exactly seventeen wallpaper groups [6]. In this paper, we use their IUC symbols: $p1$, $p2$, pm, pg, cm, pmm, pmg, pgg, cmm, $p4$, $p4m$, $p4g$, $p3$, $p3m1$, $p31m$, $p6$, $p6m$.

Several references show how to analyze a pattern and classify its frieze or wallpaper group. In this paper, we generally use the (identical) flowcharts in [6, 11] to check our analysis.

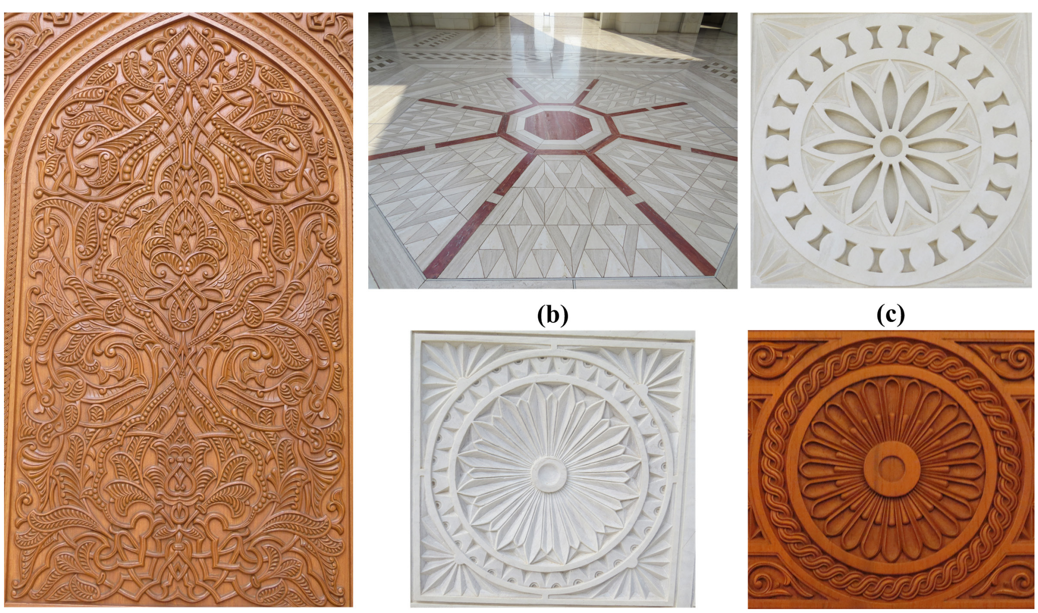
If an Islamic pattern has symmetries and is colored, we can analyze the pattern using two groups. Let G be the symmetry group of the uncolored pattern while K is the group of isometries that fix the colors in the colored pattern. In other words, K is the symmetry group of the colored pattern. It can be shown that K

$\leq G$. When all of the symmetries fix the colors, $K = G$. In this paper, we will provide the group G for each Islamic pattern analyzed. If some symmetries interchange the colors, we will also provide K.

We analyze an Islamic pattern as having a cyclic or dihedral symmetry group if the pattern is not translated. Usually, these patterns are found in arch corners or as center pieces of an elaborate pattern. An Islamic pattern has a frieze symmetry group if the motif is repeated at least three times in one direction, while an Islamic pattern has a wallpaper symmetry group if the motif is repeated at least three times in two directions. In both cases, we assume that the pattern is infinitely repeated in the correct direction. We do not restrict our analysis on the size of the patterns as some of the motifs are small and easily repeated in a small space. Arabic calligraphies of texts from the Holy Quran regularly adorn the walls of the grand mosque. Since these calligraphies are words, we ignore these in our analysis. Lastly, the authors of this paper are all male and the grand mosque is a functioning mosque. Thus, analysis of the interior of the women's prayer hall has not been done. The figures presented in this paper are representatives of the symmetry groups found in the grand mosque. There are more patterns to be admired at the mosque which are not shown here.

Patterns with Dihedral Groups as Symmetry Groups

The grand mosque exhibits at least five types of patterns with a finite symmetry group. These are D_1, D_8, D_{12}, D_{16}, and Z_4. A finite pattern with only one reflection has symmetry group of type D_1, as seen in Figure 2(a). Figure 2(b) shows a finite pattern with an 8-fold rotation and eight axes of reflection passing through the center of rotation. This pattern has symmetry group D_8. Figures 2(c)-2(e) needs a little more analysis as we see an inner pattern and an outer pattern. In Figure 2(c), the inner floral pattern has symmetry group D_{12} while the outer annulus has symmetry group D_{24}. Taking the intersection of both groups, the overall symmetry group is D_{12}. In Figure 2(d), the inner floral pattern has symmetry group D_{16} while the outer annulus has symmetry group D_{32}, making the overall symmetry group to be D_{16}. The inner floral pattern in Figure 2(e) has symmetry group D_{24} and the outer annulus has symmetry group Z_{44}, making the overall symmetry group to be Z_4.

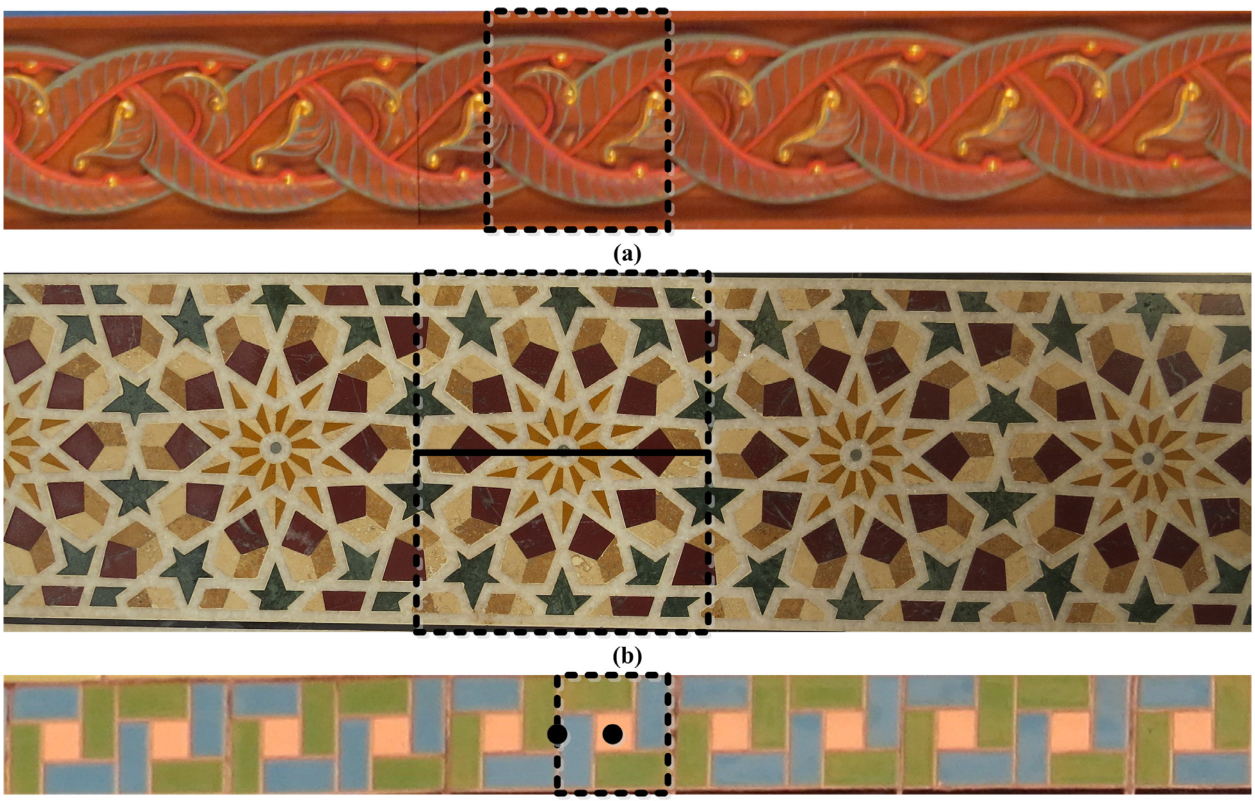


(a) **(b)** **(c)** **(d)** **(e)**

Figure 2: *Finite patterns with dihedral and cyclic groups as symmetry groups: (a) has symmetry group D_1; (b) has symmetry group D_8; (c) has symmetry group D_{12}; (d) has symmetry group D_{16}; and (e) has symmetry group Z_4.*

Patterns with Frieze Groups as Symmetry Groups

Patterns with No Vertical Reflection. There are four types of patterns with no vertical reflection, and they are denoted by *p111*, *p1a1*, *p1m1*, *p112*. Only *p1a1* has not been found so far in the grand mosque. Figure 3(a) shows a pattern with no vertical and no horizontal reflections. This pattern is only translated in one direction. The unit cell being translated is indicated by a dashed rectangle, giving the pattern a symmetry group *p111*. Figure 3(b) shows a pattern with symmetry group *p1m1*. The pattern has no vertical reflection but has a horizontal reflection. The unit cell being translated is indicated by a dashed rectangle while the horizontal reflection has axis indicated by a solid line in the middle of the unit cell. The horizontal reflection interchanges the two shades of brown in the pattern, making $K = p112$. This is our first encounter with a pattern in the grand mosque where $K < G$. Figure 3(c) is an interesting pattern. The pattern has no vertical and no horizontal reflection, but has two kinds of two-fold rotations, one centered at the center of the indicated unit cell and the other centered at the edge of the indicated unit cell (centers of rotations are at the circles). Thus, $G = p112$. However, the two-fold rotation centered at the edge of the unit cell does not fix the colors while the two-fold rotation centered at the center of the unit cell fixes the colors, making $K = p112$. In this case, K is isomorphic to G even though $K < G$.



(a) (b) (c)

Figure 3: *Frieze patterns with no vertical reflections: (a) has symmetry group p111; (b) has symmetry group p1m1; and (c) has symmetry group p112.*

Patterns with Vertical Reflection. The three types of patterns with vertical reflection are denoted by *pm11*, *pma2*, *pmm2*. Figure 4(a) shows a pattern with a vertical reflection but no horizontal reflection and no rotations. The reflection axes are on the sides and center of the unit cell, indicated by a solid line. The symmetry group of this pattern is *pm11*. Figure 4(b) shows a pattern with a vertical reflection and a two-fold rotation. The reflection axes are on the sides and center of the unit cell. The centers of rotation are inside the unit cell, indicated by a circle. This pattern has no horizontal reflection and has symmetry group *pma2*. Lastly, Figure 4(c) shows a pattern with symmetry group *pmm2*. It has vertical and horizontal reflection, and rotations, as shown on the unit cell.

(a) **(b)** **(c)**

Figure 4: *Frieze patterns with vertical reflections: (a) has symmetry group pm11; (b) has symmetry group pma2; and (c) has symmetry group pmm2.*

Patterns with Wallpaper Groups as Symmetry Groups

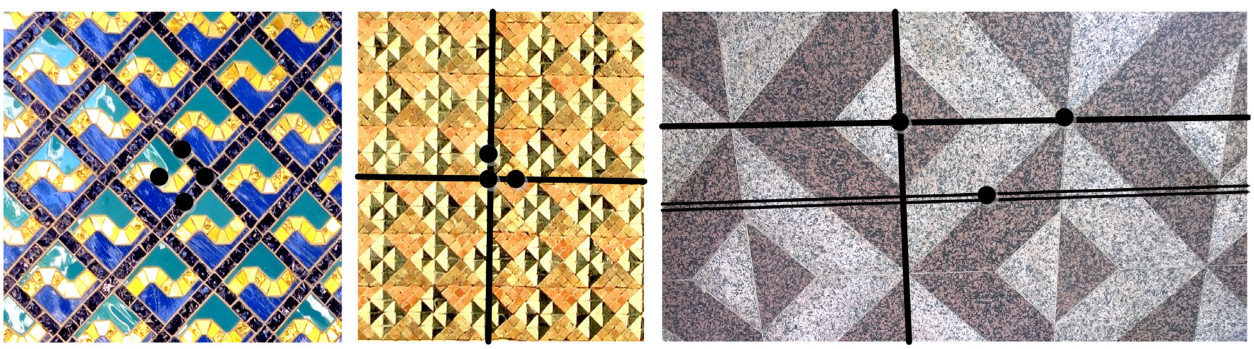
Patterns with No Rotational Symmetry. There are four types of patterns with no rotational symmetry, and they are denoted by *p*1, *pm*, *cm*, *pg*. We found two of these at the grand mosque: *pm*, *cm*. Figure 5(a) shows a pattern whose symmetry group is $G = pm$. Aside from translations, the pattern has a vertical reflection, indicated by a solid line. This pattern is particularly interesting because the cells have two different types of ornaments, making the symmetry group *pm*. On the other hand, the vertical reflection does not fix the colors. This makes the symmetry group of the colored pattern $K = p1$. Figure 5(b) shows a pattern whose symmetry group is *cm*. The pattern has a vertical reflection (single line) and a vertical glide reflection whose glide-axis is not a reflection axis (double line).

Patterns with Two-fold Rotational Symmetry. There are five types of patterns with two-fold rotational symmetry: *pmm*, *cmm*, *pmg*, *pgg*, *p*2. Only *pmm*, *cmm* and *p2* are found at the grand mosque. Figure 6(a) shows a pattern whose symmetry group is $G = p2$. The pattern has no reflections and has two-fold rotations, whose centers are on the indicated circles. These two-fold rotations do not fix the two shades of blue, which makes $K = p1$. In Figure 6(b), the pattern has two-fold rotations with centers on reflection axes and reflection axes in two directions, giving us $G = pmm$. The pattern, however, has four colors. The two-fold rotations and all reflections do not fix the colors, making the symmetry group of the colored pattern $K = pm$. The two shades of brown, however, may sometimes be indistinguishable. In the case where the browns are indistinguishable, K becomes $p1$. Lastly, Figure 6(c) has symmetry group $G = cmm$.

As compared to Figure 6(b), the pattern in Figure 6(c) has centers of rotations not in a reflection axis, indicated by a circle on a double line. On the other hand, the vertical reflection (single line) does not fix the colors while a glide reflection (double line) fixes the colors, making $K = cm$.

(a) **(b)**

Figure 5: *Wallpaper patterns with no rotational symmetry: (a) has symmetry group pm; (b) has symmetry group cm.*



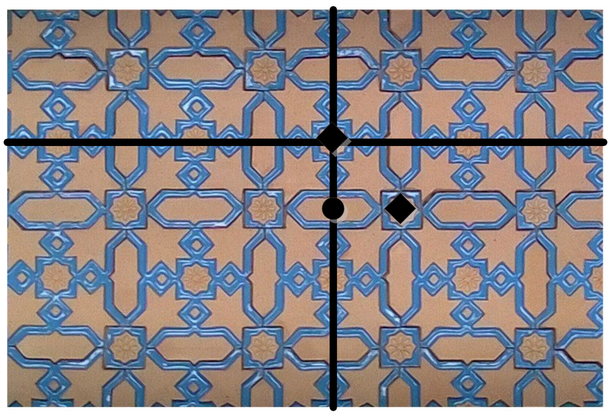
(a) **(b)** **(c)**

Figure 6: *Wallpaper patterns with two-fold rotational symmetry: (a) has symmetry group p2; (b) has symmetry group pmm; and (c) has symmetry group cmm.*

Patterns with Four-fold Rotational Symmetry. The three types of patterns with four-fold rotational symmetries are all found at the grand mosque: $p4$, $p4m$, $p4g$. Figure 7(a) shows a pattern whose symmetry group is $p4$. The black rectangular tiles are arranged in such a way that no reflections can occur in the entire pattern. The centers of the four-fold and two-fold rotations are indicated by a diamond and a circle, respectively. Figure 7(b) shows a pattern whose symmetry group is $p4m$. The pattern has four-fold rotations and reflections with axes in four directions. Two of the reflection axes are indicated by solid lines while centers of rotations are indicated by either a diamond or a circle. The symmetry group of the uncolored pattern in Figure 7(c) is $G = p4g$. The pattern has four-fold rotations and reflections with axes in two directions, indicated in the figure accordingly. This pattern has two colors. The four-fold rotations do not fix the colors but the two-fold rotations do. The vertical reflections also fix the colors, making the symmetry group of the colored pattern $K = cmm$.

Patterns with Three-fold and Six-fold Rotational Symmetries. There are three types of patterns with three-fold rotational symmetry: $p3$, $p3m1$, $p31m$. There are two types of patterns with six-fold rotational symmetry: $p6$ and $p6m$. None of these were found at the grand mosque.

(a)



(b)

(c)

Figure 7: *Wallpaper patterns with four-fold rotational symmetry: (a) has symmetry group p4; (b) has symmetry group p4m; and (c) has symmetry group p4g.*

Summary and Further Work

In this paper we observed the following groups as symmetry groups of various patterns found in the Sultan Qaboos Grand Mosque: D_1, D_8, D_{12}, D_{16}, Z_4 for finite patterns, *p111*, *p1m1*, *p112*, *pm11*, *pma2*, *pmm2* for the frieze groups, and *pm*, *cm*, *pmm*, *cmm*, *p2*, *p*4, *p*4*m*, *p*4*g* for the wallpaper groups. For infinite patterns, orders of rotations are limited by the possible frieze and wallpaper groups. If we include both the uncolored symmetry groups and the color fixing groups, the tilings here demonstrate six of the seven frieze groups and nine of the twelve "rectangular" wallpaper groups (those without 3-fold or 6-fold rotations). For finite patterns, however, there is no limitation on the possible orders of rotations. Interestingly, we only see 4-fold, 8-fold, 12-fold, and 16-fold rotations in symmetry groups for finite patterns.

While Oman is geographically and, to some extent, theologically separated from other denominations of Islam, we can see that it has the same interest in incorporating a wide variety of mathematical forms of symmetry. Of course the grand mosque was built between 1995 and 2001, and it was intended to "present the most varied Islamic traditions," so this is somewhat expected. Further work with more historic sites could give substantial insight into whether this is a historic phenomenon or a more recent phenomenon.

It is also interesting to note that we did not find any 3-fold and 6-fold rotations, resulting in no patterns with symmetry groups $p3$, $p3m1$, $p31m$, $p6$ and $p6m$. Furthermore, the three missing rectangular patterns are *pg*, *pmg*, and *pgg*; and the missing frieze pattern is *p1a1*, all being designs that focus on glide reflections. Some of these designs which do occur, specifically *pma2* (frieze), *cm* and *cmm* (wallpaper), do contain glide reflections, but some of these can occur "accidentally" as a consequence of other intended symmetries. Again, further work could look at whether these symmetry preferences are historic or modern.

Aside from mosques, Islamic patterns can be found in forts, souqs (traditional markets), doors and gates, and other building complexes. Islamic patterns in Oman can be found in both functional and non-functional building complexes. Analyzing Oman's cultural properties is new [3], and more work can be done by analyzing these patterns using group theory.

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