Sangaku--Japanese Mathematics and Art in the 18th,19th and 20th Centuries

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Abstract

In the 18th, 19th, and 20th centuries ordinary people enjoyed traditional Japanese mathematics all over Japan. They made *sangaku*, votive wooden tablets with geometry problems, and hung them in many temples and shrines. The problems were presented artistically in color to attract the visitor's eye. The world of sangaku means both Japanese mathematics and art dedicated to temples and shrines.

1 *Sangaku* as Sacred Mathematics

From the 18th to early 20th centuries, Japanese mathematicians consisting of professionals, amateurs, women and children created *sangaku,* which are wooden tablets adorned with beautiful geometric problems, presented as works of art. The name literally means mathematical tablet (*san* = *mathematics*, *gaku* = *tablet*). The creators of these sangaku hung them by the thousands in Buddhist temples and Shinto shrines throughout Japan. For that reason the entire collection of sangaku problems has come to be known as geometry of sacred mathematics [1], [2], [3]. Figure 1 shows a wooden tablet that was created and hung in 1875 under the roof of the Kaizu Tenman shrine located at Shiga prefecture.

Figure 1: *A wooden tablet under the roof of the Kaizu Tenman shrine.*

2 Exhibition of *sankagu* in 2004 at the Nagoya Science Museum

Fukagawa supervised the first exhibition of sangaku in 2004 at the Nagoya Science Museum with about one hundred sangaku transported from all over Japan. Figure 2 shows one of the largest sangaku from Yamagata prefecture; it is about 151 cm high and 453 cm wide. This votive tablet was originally hung in 1823.

Figure 2: *One of the largest sangaku from Yamagata prefecture*

Figure 3 shows a picture of one of the most beautiful sangaku in a frame carved and decorated with flowers; it is 82 cm high and 173 cm wide. It was hung in 1854 at the Sugawara Tenman shrine of the Mie prefecture. This sangaku first appeared in print in 2005 in a book about the history of mathematics by R.L.Cooke [4].

Figure 3: *One of the most artistic sangaku*

Figure 4 shows a faithful replica (made with Fukagawa's help) of a sangaku originally created and hung at the Atsuta shrine in Nagoya city in 1844 . The original tablet, which was 60 cm high and 240cm wide, was unfortunately lost. The same shrine also made a replica of a sangaku wooden tablet 30 cm high and 40 cm wide that had originally been hung in 1841 (see Figure 5). Both of these sangaku display famous theorems about tangent (or "kissing") circles, a favorite topic. Figure 5 shows the beautiful theorem relating the radii of four circles tangent to two given circles:

$$\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{c}}=\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{d}}$$

where the radii correspond to colored circles as follows: a (dark green, lower right), b (light green, lower left), c (dark blue, upper left), and d (orange, upper right). However, there are no details about the solution to this theorem shown on the tablet (likely because a long calculation is needed).

Figure 4: *Replica of a sangaku hung at the Atsuta shrine*

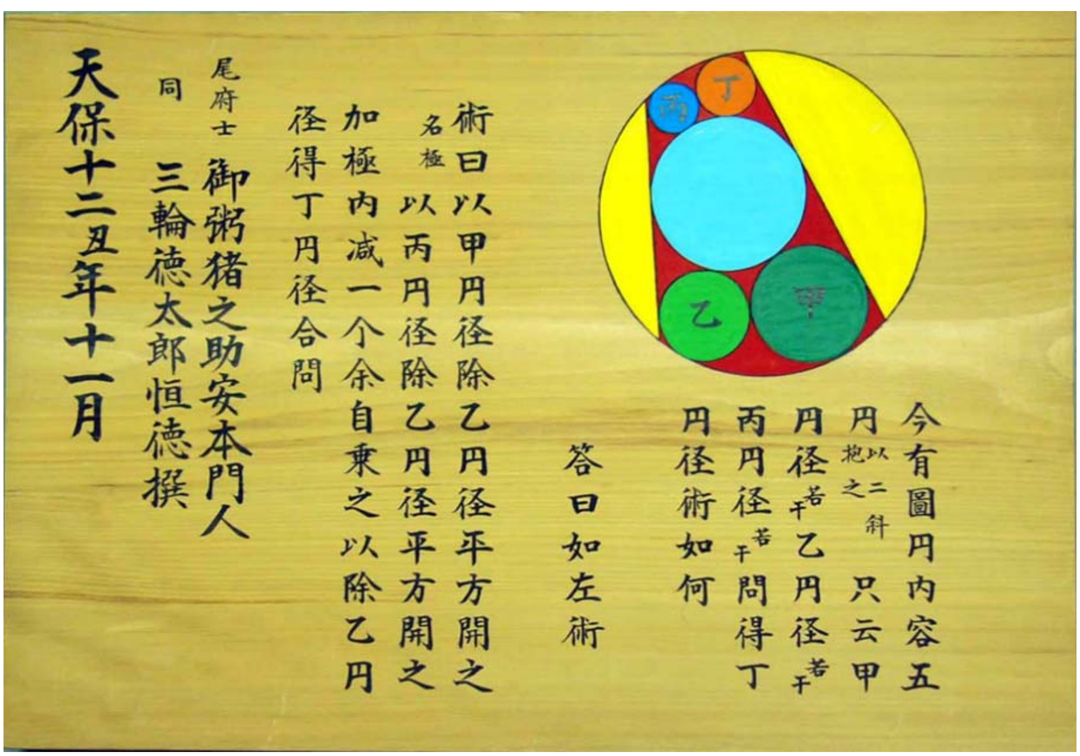


Figure 5: *A theorem about kissing circles*

Since the exhibition in 2004 and the publication in English of our book on sangaku in 2008 [3], many mathematicians from all over the world have come to visit Japan to look at these sangaku tablets. A Danish group visited the Myoujyourin-ji Temple in Gifu prefecture in 2012 to see an original sangaku that has survived (Figure 6).

Figure 6: *A Danish group views an original sangaku at the Myouiyourin-ji Temple*

There are many interesting sangaku that could not be transported to the exhibition hall of the museum because they were mounted and could not be removed or were too fragile. For instance, Figure 7 shows some sangaku that are fixed as ceiling panels in a small Itsukushima shrine in Fukushima prefecture. The center panel shows squares and triangles that outline a house-like figure inscribed in a semicircle. It states the following relation; $c = (1 - \sqrt[3]{0.25})a$ where c and a are the lengths of the sides of the small white squares and the orange squares, respectively.

Figure 7: *Sangaku mounted as ceiling panels*

In the same shrine another ceiling panel shows a beautiful painting along with a mathematical problem (Figure 8). This gives a perfect example of "Art and Mathematics." The snake entwining the tree is "Art" and the writing in Japanese below it is "Mathematics". The problem stated here is:
A snake 2 meters long is sleeping; it is wrapped once around a circular branch and the parts of the snake not wrapped around the branch have total length equal to the diameter x of the branch. Find the diameter x. The answer is $x + \pi x = 2$, or $x = 2/(1 + \pi)$.

Figure 8: *An artistic illustration of a geometry problem*

3. Sangaku as Art

In this section, we show some beautiful sangaku which are mainly artistic works rather than mathematics. The sangaku with a beautiful carved frame in Figure 9 was originally hung in 1873 at Katayamahiko shrine in Okayama prefecture; it was dedicated by an aged rich trader who lived near the shrine. This tablet, which is 88 cm high and 162 cm wide, has survived after 140 years.

Figure 9: *An artistic sangaku that has survived 140 years*

The sangaku in Figure 10 depicts a spring festival under cherry blossoms, men and women enjoying rice wine and food. The tablet also gives three mathematics problems on its left side. One of them goes like this: The total number of men and women is 307, however there are 3 more men than women. Find x, the number of men. The solution is: $x + (x - 3) = 307$, so x = 155.

Figure 10: *A village scene with three mathematical problems*

4: Difficult theorems in *Sangaku*

Some sangaku pose difficult theorems. For instance, Descartes' circle theorem, stated in 1643, says that given four mutually tangent circles with radii r_i and where the "bend" k_i of circles i is defined as $k_i = 1/r_i$ then the following equation is true:

$$(k_1 + k_2 + k_3 + k_4)^2 = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2)\,.$$

Hence, given three mutually tangent circles, it is always possible to construct a fourth circle tangent to the other circle. This theorem was discovered independently by a Japanese mathematician. Figurer 11 shows the theorem as it appeared in an old Japanese text "Sanpo Tenzan Syogakusyou" published in 1830, and Figure 12 shows one page from Descartes' work.

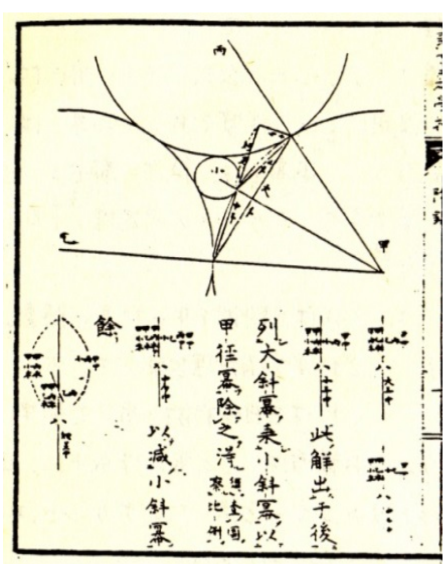
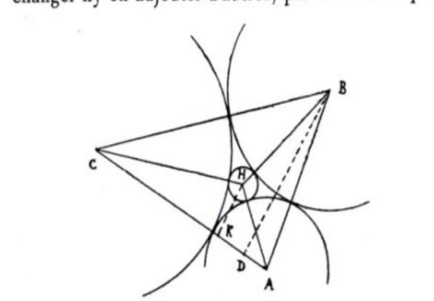


Figure 11: *Descartes' Theorem in a Japanese text*



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trois AH, BH & CH, ce que n'ont pas les premieres. Et en ſuiuant le calcul auec ces ſix lettres, ſans les changer ny en adjoûter d'autres, par le chemin qu'a

pris Voſtre Alteſſe (car il eſt meilleur, pour cela, que celuy que i'auois propoſé), on doit venir à vne équation fort reguliere, & qui fournira vn Theoreme aſſez court. Car les trois lettres *a, b, c,* y ſont diſpoſées en meſme façon, & auſſi les trois *d, e, f.*

Figure 12: *Original Descartes' theorem*

When we extend Descartes' circle theorem to 3-dimension space, we get Soddy's famous "Hexlet" theorem on kissing spheres, first published in the journal *Nature* in 1936 [5]. This says that given any three mutually tangent spheres, you can always find a chain of six other spheres that are tangent to the given ones, and in the chain, each sphere is tangent to two others. Yet this same theorem was shown in a sangaku tablet hung in 1822 at the Samukawa shrine in Kanagawa prefecture, more than one hundred years before Soddy's discovery. This sangaku was made by Irisawa Hiroatsu in the family of Uchida Itsumi. The original tablet was lost but it was recorded in Uchida's 1832 book *Kokon Sankan*, and in 2009 a replica was made with Fukagawa's help, based on this record and dedicated to the Hōtoku Museum in the Samukawa Shrine. See Figure 13.

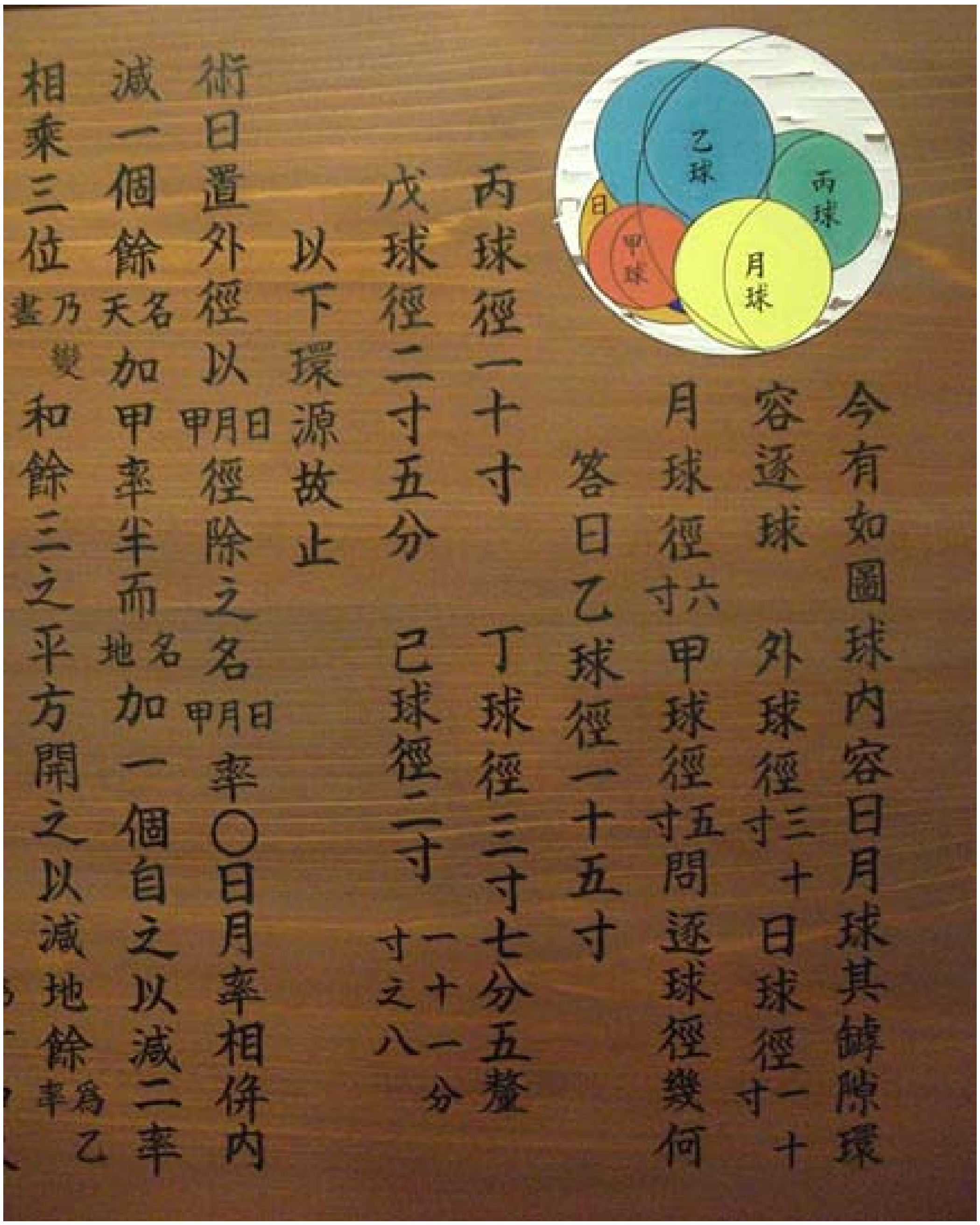

Figure 13: *Replica of 1822 sangaku in the Hōtoku museum in Samukawa Shrine.*

In closing, it is important to note that in the Edo era of the 18^{th} and 19^{th} centuries in Japan, ordinary people enjoyed mathematics in daily life, not as a professional study but rather as an intellectual popular game and a recreational activity. This is depicted in the large sangaku tablet (93 cm high and 170 cm wide) first hung in 1861 at the Souzume shrine of Okayama prefecture (Figure 14).

Figure 14: *A sangaku depicting the popularity of mathematics as a game*

Even today, many of these sangaku problems are a source of pleasure and challenge. There are many websites that contain summaries of the history of sangaku and pose several of the problems found in sangaku. Some provide solutions given by Fukagawa, Pedoe, Rigby, or Rothman [1], [2], [3] and others give original solutions; many solutions are “left to the reader.” A few of these sites are listed in the References [6], [7], [8].

References

[1] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems,* Winnipeg, Canada: The Charles Babbage Research Centre (1989).

[2] H. Fukagawa and J.F. Rigby, *Traditional Japanese mathematics problems of the 18th and 19th Centuries*, SCT Press, Singapore (2002).

[3] H. Fukagawa, T. Rothman, *Sacred Mathematics: Japanese Temple Geometry,* Princeton University Press (2008).

[4] R. L.Cooke, *The History of Mathematics: A Brief Course,* Wiley-Interscience (2005).

[5] http://en.wikipedia.org/wiki/Soddy’s_hexlet

[6] http://people.cohums.ohio-state.edu/unger26/Sangaku.pdf

[7] http://files.eric.ed.gov/fulltext/EJ720042.pdf

[8] http://www.cut-the-knot.org/pythagoras/Sangaku.shtml