Removing Tremas with a Rational Function

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Abstract

In his charming inimitable style, Benoit Mandelbrot, in his book *The Fractal Geometry of Nature*, describes the removal of “tremas”. Two examples of figures constructed by removing tremas are familiar fractals: carpets and gaskets. This paper will briefly describe the difference between a carpet and a gasket and then illustrate how the Julia set of a rational function can be a carpet, a gasket or another structure generated by “removing tremas”. An added bonus is beautiful images resulting from an imaginative assignment of color.

Introduction

In his famous book, *The Fractal Geometry of Nature* [4], Mandelbrot begins his discussion of *removing tremas* by describing the familiar Cantor Middle-thirds Set: “The middle-third portion cut out of (the interval) [0,1] to form a gap is henceforth denoted as ‘trema generator’ from the Greek *τρημα* meaning *hole*.” Later in the book he discusses the cutting out of tremas from two and three dimensional spaces; the results are: galaxy clusters, Sierpinski and Apollonian Gaskets, Sierpinski Carpets, Menger Sponges, lunar craters and disk tremas. I will discuss briefly the topological difference between gaskets and carpets and show some traditional ways of generating carpets and gaskets. My goal is to illustrate the beauty and variety of the Julia sets obtained by iterating a family of rational functions having a single pole and defined by one complex parameter. By varying the parameter, a variety of attractive carpets, gaskets, images that resemble galaxy clusters and other figures obtained by the “removal of tremas” can be created.

Carpets and Gaskets

The Sierpinski Carpet, Figure 1(*a*), is constructed by removing open squares from an original square; the Sierpinski Triangle, Figure 1(*b*), is constructed by removing open triangles from an original triangle. Both sets are compact, connected and path connected. The topological difference is the existence of “local cut points” in the gasket. Let C be the original square from which we construct the carpet and let C_n be the set of points remaining in C after the nth stage of removing squares. Let G be the original triangle from which we construct the gasket, and let G_n be the set of points remaining in G after the nth stage of removing triangles. In Figure 1 the regions colored black are those that remain after the removal of the white squares or triangles. At stage n in the construction of the carpet, for any open set U in C_n and any two points p and q in U, there exists a path in U from p to q. At any stage in the construction of the gasket, for any open set V in G_n and any two points r and s in V there exists a path in V from r to s. If

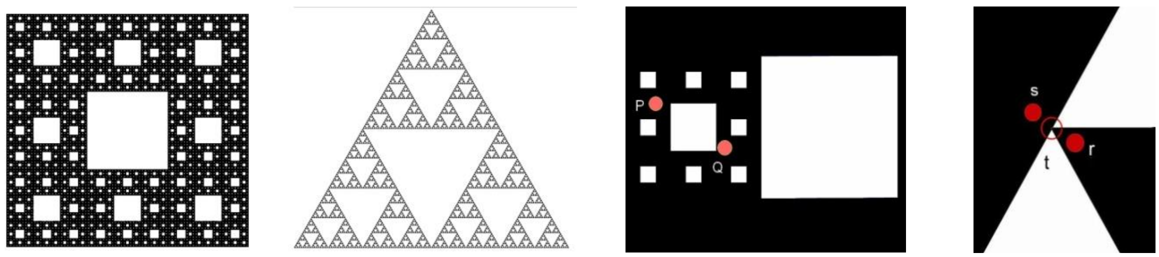

Figure 1: *(a) Sierpinski Carpet (b) Sierpinski Gasket (c) no cutpoints (d) a cutpoint*

any third point u is removed from C_n there is still a path in C_n from p to q. (Figure 1(c)). However in G_n, if we remove a point t that is on the boundary of two removed triangles there exists an open set V in G_n (the two small white areas inside the circle in Figure 1(d) that is not path-connected. The point t is called a *local cut point*. It disconnects the two white regions. In the limiting case the set G contains local cutpoints while the set C does not. [3]. The difference between a carpet and a gasket is the existence of local cut points in a gasket. Figure 2, Mandelbrot's construction of the gasket, illustrates the fact that G can be defined as the limit set of a sequence of one-dimensional curves.

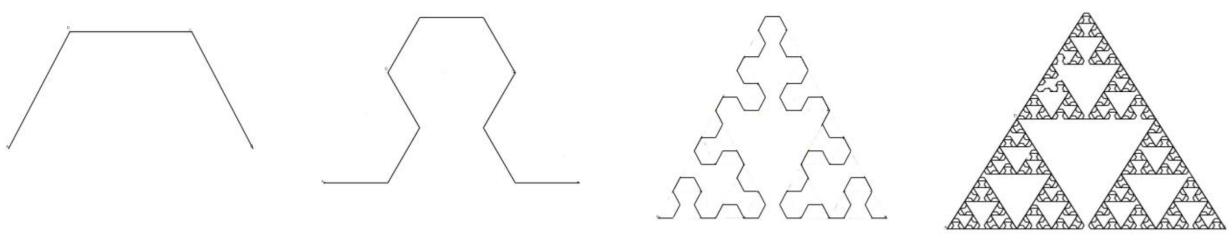

Figure 2: *Mandelbrot's construction of the gasket*
(a) Initiator (b) Generator (c) After 3 stages (d) After 6 stages

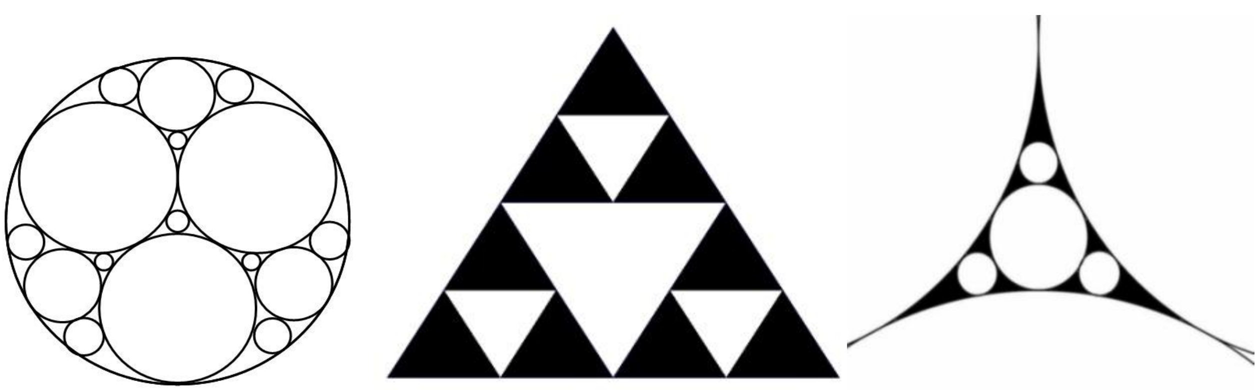

Figure 3: *(a) Another famous gasket, The Apollonian Gasket. (b) and (c) the Sierpinski Gasket is homeomorphic to the Apollonian Gasket.*

Notice the cutpoints in another famous gasket, the Apollonian Gasket in Figure 3(a). In *The Fractal Geometry of Nature*, Mandelbrot describes how Apollonius of Perga, around 200 BC, discovered an algorithm to draw the circle tangent to three given circles. Figure 3(b and c) is intended to be a visual argument that the Apollonian Gasket is homeomorphic to the Sierpinski Gasket.

Cutting out tremas with a rational function

Consider the rational function of a complex variable with one parameter and a pole at z = 0, $f(z) = z^2 + c/z^2$. Then $f(0) = \infty$. First let c be very small in magnitude, for example, c = -.05. Start with a point z_0 and find its orbit under f: $\{z_0, f(z_0), f(f(z_0), ...\}$. If the initial point, z_0 has magnitude greater than 1.5, say, it is easy to see that the orbit is unbounded and lim $f(f(...f(z_0))) = \infty$. If the initial point z_0 is very small in magnitude, then $|f(z_0)|$ will be very large (in particular, greater than 1.5). In Figure 4(a) there are two white regions, which we will call **B** (the region outside the black disk) and **T** (the region inside the disk). **B** is the set of points z such that $|f(z)| > 1.5$ and **T** is the set of points not in **B** such that $|f(f(z)| > 1.5$. **T** is f^{-1}(**B**). In Figure 4(b), in addition to **B** and **T** the four pre-images of **T** under f are shown in white. In Figure 4(c) the four pre- images of each of the four pre-images are also in white, and Figures 4(d) and 4(e) show two more stages of backwards iteration. The process is reminiscent of the way in which the Sierpinski carpet is constructed.

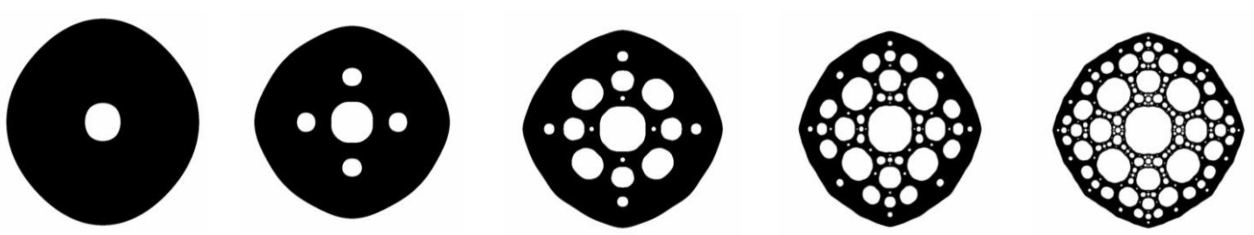

Figure 4: *The first five backward iterations, f^{-1}(**B**) for c = -.05*

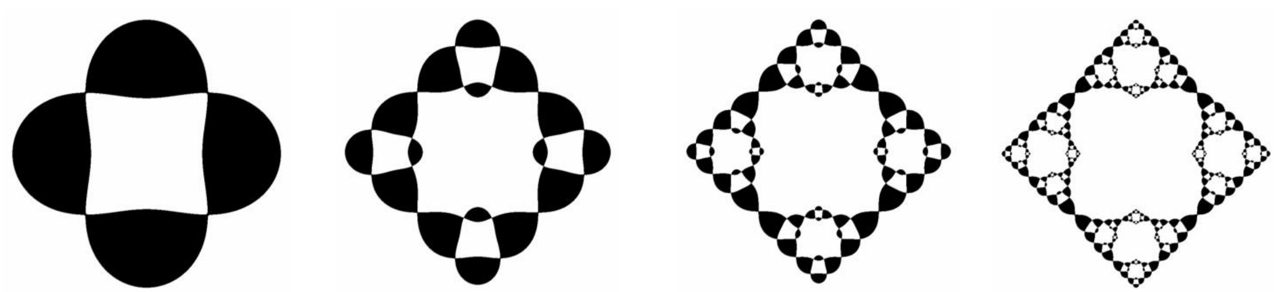

Figure 5: *The first four backward iterations f^{-1}(**B**) for c = -.36428*

Now change the parameter c. Figure 5 shows the first four backward iterations for c = -.36428. What do you see emerging? A gasket! Notice the cutpoints where **T** meets **B**. Compare Figure 4 and Figure 5 and notice the way the removed tremas meet the boundary in Figure 5 just as they do in the Sierpinski Gasket. The Julia Set of each function is the limiting figure. The value of the parameter c determines the Julia Set. For c = -.05 the Julia Set is a carpet and for c approximately equal to -.36428 the Julia Set is a gasket! Robert Devaney, on his web page [2], has made available numerous papers which he co-authored with others and that deal with the dynamics of rational functions of the form $f(z) = z^n + c/z^m$ for z in the complex plane and c a complex parameter. The simplest case is the case considered here, when $n = m = 2$. As illustrated there are values of the parameter c for which the Julia Set of f is a Sierpinski Gasket while for other values of c the Julia Set is a Sierpinski Carpet. [1]. Other values of c, while not yielding a carpet or a gasket, can be used to produce other interesting images of figures obtained by removing tremas.

Actually in Figures 4 and 5 the regions are only approximations to the **B** and **T** in the literature. For a fixed value of c, **B** is called the immediate basin of attraction of ∞ and **T** is the open set containing the origin such that if z is in **T** (but not in **B**) then $f(z)$ is in **B**. If $c = 0$ then $f(z) = z^n$ and the dynamics of f are well known: there are two attracting fixed points, 0 and ∞, and the Julia Set is the unit circle, on which the dynamics are chaotic. For c not zero, the situation changes drastically since then $f(0) = \infty$. For a more rigorous definition of **B** and **T**, see [1] or [2]. If the critical points of f lie in **B** then the Julia Set is a Cantor set. [1] If they are not in **B**, then there is an open set **T** containing the origin such that if z is in this set, $f(z)$ is in **B**. (**T** stands for trap door since if a point on an orbit enters **T**, the next point in the orbit will be in **B**. The boundary of **B** and the boundary of **T** both belong to the Julia Set.

For the case where $n = m = 2$ it is easy to check that there are four critical points which are the fourth roots of c; also there are four points, the fourth roots of $-c$, whose image under f is zero. For each of the fourth roots of $-c$ there is an open set containing that root whose image under f is **T**. Figure 6(a) is a rough diagram showing the circle $|z| = c^{1/4}$ for a "small" real positive value of c. The four black points on the circle are the critical points and the interiors of the four discs centered on the critical circle are the pre-images of **T**. **T** and its preimages are actually more irregularly shaped and the shape depends on c allowing us to get interesting images.

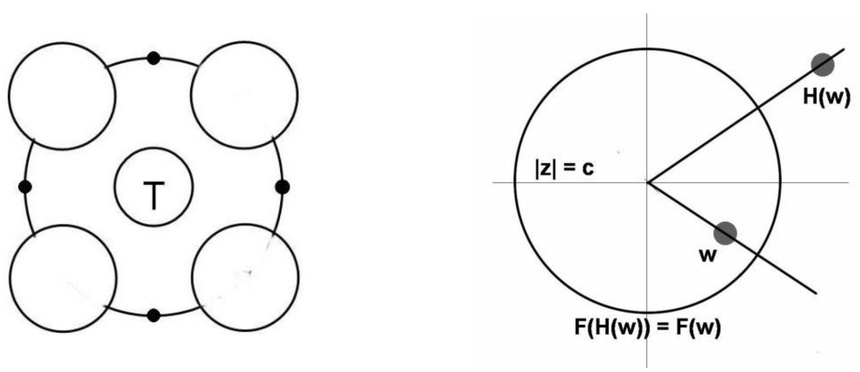

Figure 6: *(a) The circle $|z| = c^{1/4}$ for a real value of c   (b) The involution* ***H***

This circle $z = |c|^{1/4}$ is called the *critical circle*. The key to understanding everything is the involution $H(z)=\sqrt{c}/z$ which maps the outside of the circle to the inside, and the inside to the outside, and it maps

B to **T** and **T** to **B**. And, most importantly, $f(H(z)) = f(z)$. Figure 7 is a sequence of images showing the critical circle in white for real-valued c as c increases along the real axis from 1/16 to 1/8. The symmetry with respect to the critical circle illustrates the fact that the involution $H(z)$ interchanges **B** and **T**. The orbits of points in the black areas remain bounded. The non-black points eventually enter **T** and then in the next iteration they enter **B**. As $|c|$ increases, the radius of the critical circle increases and

Figure 7: *The critical circle for positive real-valued c*

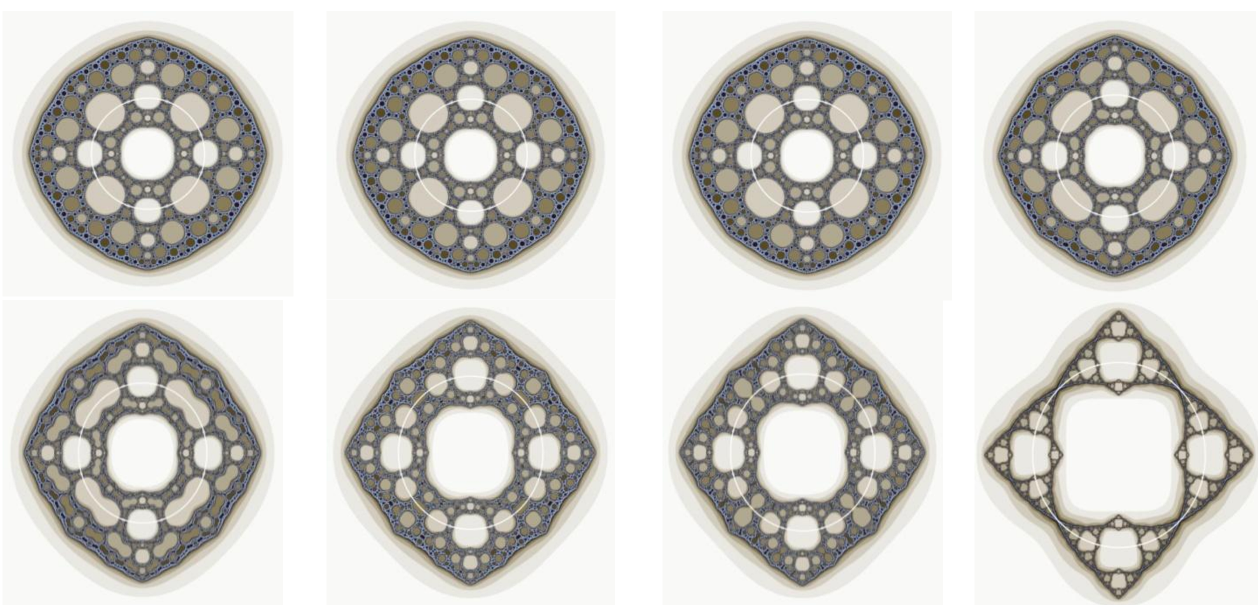

Figure 8: *Evolution of a carpet into a gasket*

the boundaries of **B** and **T** move closer together. Orbits of points in the red and blue areas escape, but they escape more slowly than the points in the white areas. Figure 8 shows the Julia Set for f with negative real-valued c taking on the values from -.065 to -.36428. At c approximately -.36428, as noted earlier, the Julia Set is a "Sierpinski Gasket" Notice that for c increasing along the positive real axis the conditions are not right for a Julia set gasket, whereas if $|c|$ increases along the negative real axis the cutpoints, where the boundary of **B** meets the boundary of **T,** emerge and a gasket does appear.

Creating artwork

Interesting pictures can be produced by iterating the function $f_c(z) = z^n + c/z^m$ for other values of n, m and c. For $n = 2$ and $m = 1$ and c approximately -.5952 another gasket appears. For other values of c attractive designs appear when each point is colored according to the number of iterations it takes for the orbit of that point to enter **B**. Figure 9 is an example; in 9(a) the large white area in the center is **T** (the trap door); It is easy to see the pre-images of **T**; they all have the same shape. Near the fixed points it takes longer and longer for points to escape; hence the spirals. Figure 9(b) shows a zooming in of a neighborhood of one of the fixed points. The image in Figure 10 resembles some of the many interesting

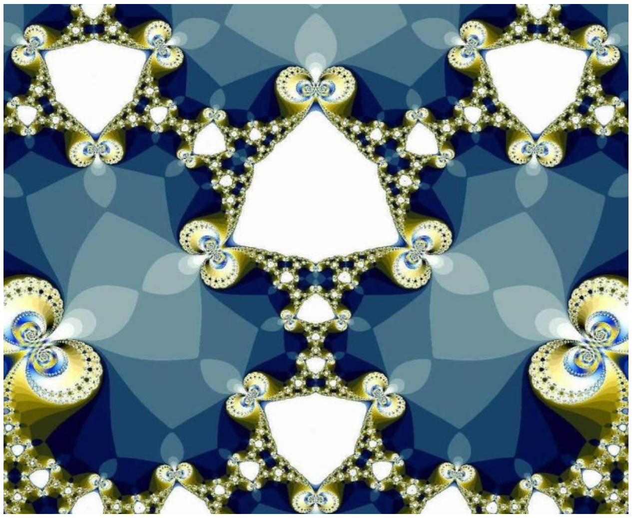
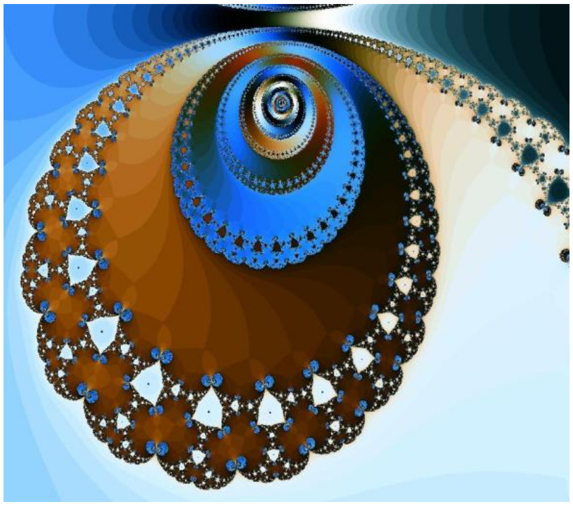

Figure 9: (a) *Escape time determines color* (b) *Zooming in on a fixed point* .

illustrations of *tremas* in Mandelbrot [4] ("Moon Craters and Galaxies"). It was produced by iterating f for a value of c for which the Julia Set is a Cantor Set; the orbit of almost every point eventually enters **B**. The color of a point is based on the number of iterations required for the orbit of the point to enter **B**. For other values of n and m interesting images can be produced. Figures 11 and 12 are the result of iterating $f(z) = z^n + c/z^m$ for $1/n + 1/m < 1$. "McMullen has shown that if $1/n + 1/m < 1$, when c is small, the Julia Set explodes from a single circle to a Cantor set of simple closed curves surrounding the origin."[1] In Figure 11(a) the larger white areas are the immediate preimages of the trap door, the gold necklaces are the preimages of the white necklaces, and points in the brown areas take longer to escape; the Julia Set is not visible but we can imagine it as snaking around the origin between the gold and white necklaces. In Figure 11(b) $n+m = 5$ and in Figure 11(c) $n+m = 6$.

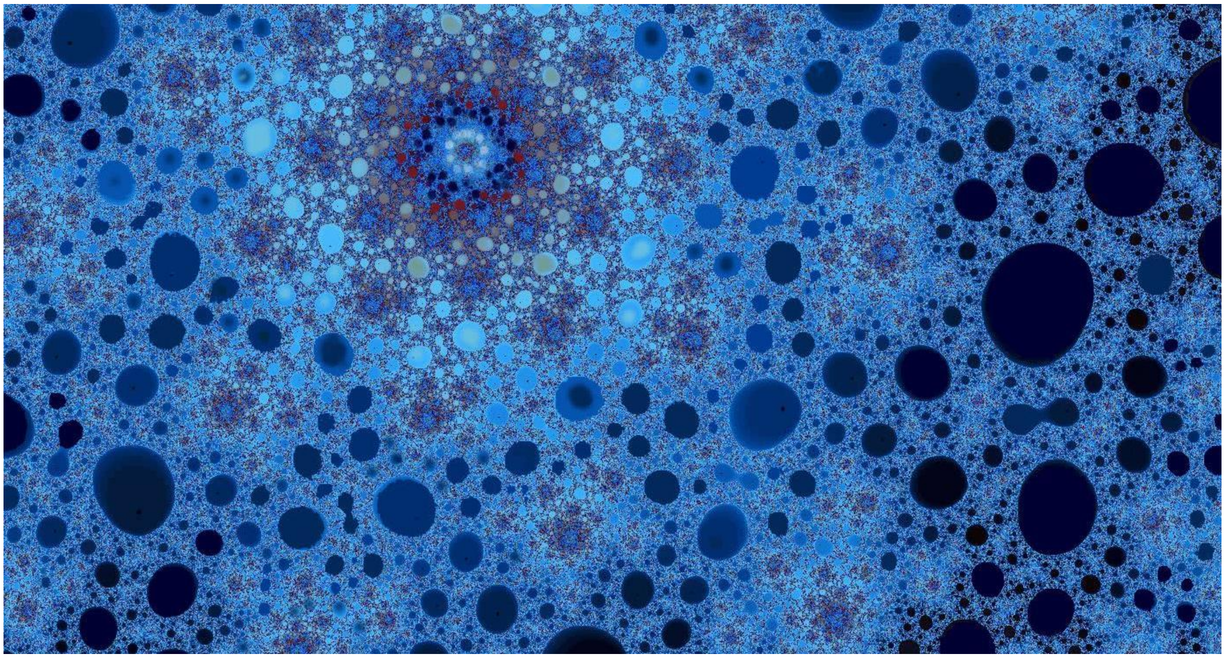

Figure 10: *Removing tremas*

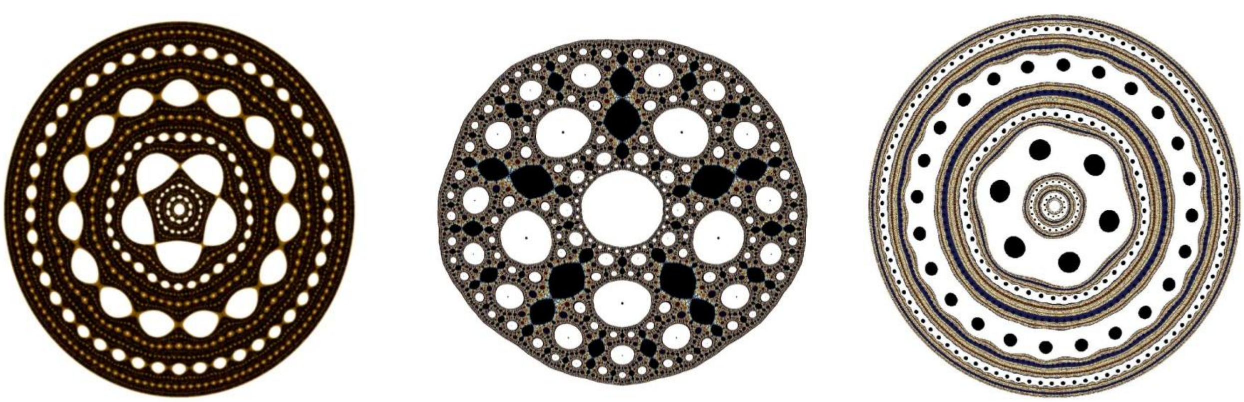

Figure 11: $f_c(z) = z^n + c/z^m$ for $1/n + 1/m < 1$ for different values of n, m and c

Future Work

This paper grew out of my interest in the topological difference between a Sierpinski Carpet and a Sierpinski Gasket and the book *Dynamics on the Riemann Sphere, A Bodil Branner Festschrift*[1]; both led me to other papers by Robert Devaney and co-authors. I have always been interested in Mandelbrot's discussion of removing tremas; iterating a complex valued function that has a pole means that the pole has a neighborhood with an infinite number of pre-images and removing the neighborhood and its pre-images is a method for removing tremas. The only functions considered in this paper are perturbations of the functions $f(z) = z^n$ whose dynamics are completely understood. These functions have a single pole at zero. There is newer research concerning the dynamics of rational functions with poles chosen to satisfy certain conditions related to the Mandelbrot Set. The mathematics is fascinating; in addition the artist can use his creativity to appeal to a much wider audience. An infinity of pictures of Julia Sets of rational functions is out there waiting for someone to discover them.

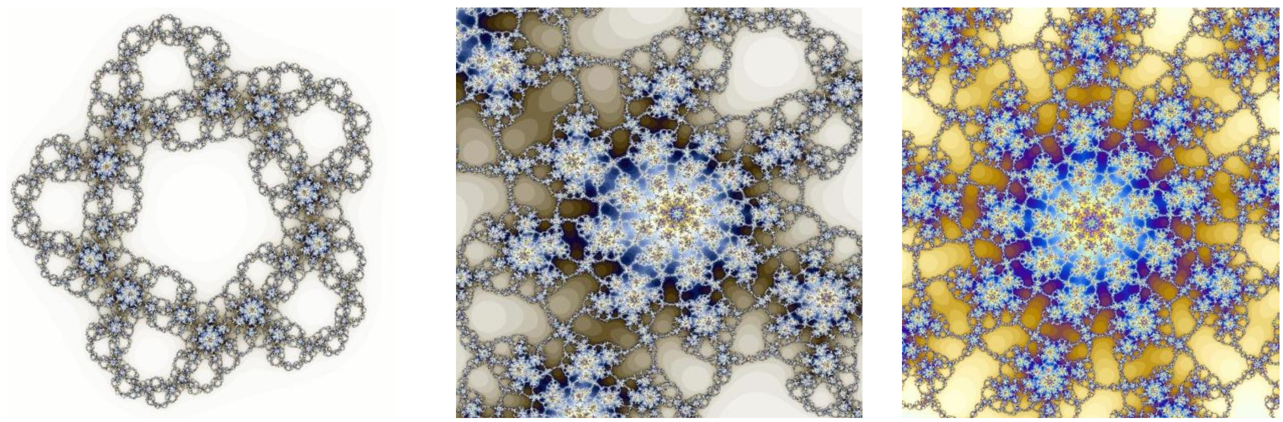

Figure 12: *Zooming in on a point*

References

[1] Paul Blanchard, Robert L. Devaney,Daniel M. Look, Pradipta Seal, Stefan Siegmund, David Uminsky. Sierpinski Carpets and Gaskets as Julia Sets of Rational Maps, pp 97-119, *Dynamics on the Riemann Sphere, A Bodil Branner Festschrift*, Paul G. Horth and Carsten Lunde Petersen, Editors, European Mathematical Society, 2006, 3-03719-011-6

[2]Robert Devaney, http://www.math.bu.edu/people/bob/papers.html

[3] Joel Louwsma. Homeomorphism Groups of the Sierpinski Carpet and Sierpinski Gasket, REU project at University of Michigan, July 1, 2004, http://www.math.lsa.umich.edu/undergrad/REU/ArchivedREUpapers/Joels%20Paper.pdf

[4] Benoit B. Mandelbrot. *The Fractal Geometry of Nature*, W.H. Freeman and Company, New York, 1983