Multidimensional Impossible Polycubes

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Abstract

We first derive a 3-dimensional impossible polycube by forcibly deforming the projection of a 3-dimensional polycube. This procedure is extended into n(≥4)-space to construct n-dimensional impossible polycubes represented in 2- or 3-space. They are useful as fundamental grid patterns for imaging various n-dimensional impossible figures in our 3-space. On 2-space, especially, each pattern can be composed of [n/2] kinds of rhombi grouped into *n* congruent periodic portions which spirally fill a semi-regular 2n-gon. The same [n/2] kinds of rhombi compose a radial quasi-periodic pattern in a regular 2n-gon which is derived from the projection of an n-dimensional polycube.

Basic n-dimensional figures in this paper

An n-dimensional impossible polycube (an impossible n-polycube) in this paper is composed of mainly two kinds of n-dimensional regular polytopes. One is the n-dimensional cube (n-cube) as the 2n-cell composed of *2n* numbers of (n-1)-cubes, while the other is the n-dimensional regular tetrahedron (n-tetrahedron) as the (n+1)-cell or n-dimensional simplex composed of *n+1* numbers of (n-1)-tetrahedra.

Several n-cubes may construct an n-polycube which is a portion of space-filling agglomeration of n-cubes. On the contrary, 3-dimensional regular tetrahedra (3-tetrahedra) can fill 3-space with 3-dimensional regular octahedra, each being the dual of a 3-cube. Buckminster Fuller's octet-truss or the polyoctet in this paper is a portion of this triangular 3-dimensional space-filling agglomeration (Figure 1).

They are usually represented by orthogonal isometric projections into 2- or 3-space in this paper.

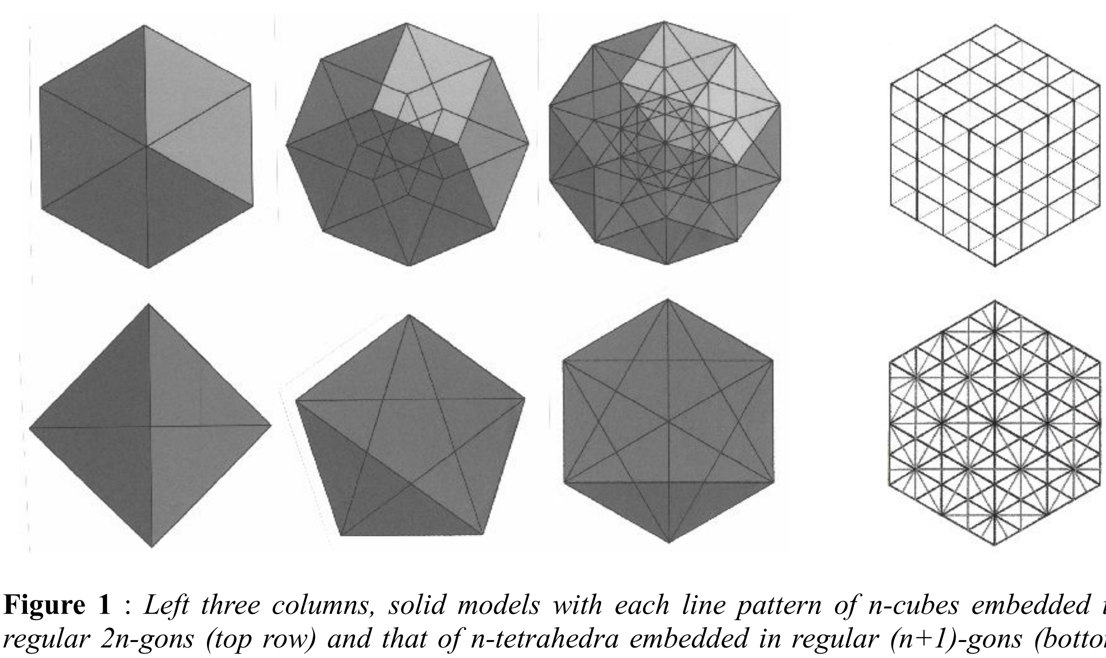


Figure 1 : *Left three columns, solid models with each line pattern of n-cubes embedded in regular 2n-gons (top row) and that of n-tetrahedra embedded in regular (n+1)-gons (bottom row). From left to right, n=3, 4, and 5. Right end column, line patterns of a 3-polycube (top) and a polyoctet (bottom). CG: M. Ishii.*

Impossible 3-polycube

An impossible 3-polycube is derived from the forcibly deformed projection onto 2-space of a 3-polycube. It is composed of overlapping congruent rhombi which are arranged in the shape of not a regular but a semi-regular hexagon having 3-fold symmetry (Figure 2, left). When the certain hidden lines are neglected, a spiral arrangement of three congruent periodic portions made of congruent rhombi appears around the central regular triangle as a 2-tetrahedron (Figure 2, right). This pattern can embed Penrose's impossible tribar which consists of three 4-cubic prisms having 3-cubes as their bases (Figure 3, left end). In other words, the impossible 3-polycube may also be constructed by contracting Penrose's tribar (Figure 3, right two).

Various 3-dimensional impossible figures may be imagined in this impossible polycube (Figure 4). A remarkable example is an impossible version of the polyoctet whose every edge coincides with a diagonal of the square face of an impossible polycube (Figure 5).

On the other hand, the projection of a 3-polycube can be composed of overlapping congruent rhombi which are arranged in the shape of not a semi-regular but a regular hexagon having 6-fold symmetry as shown in Figure 1, top right end. When the certain hidden lines are neglected, a radial arrangement of three congruent periodic portions made of congruent rhombi appears around the central point.

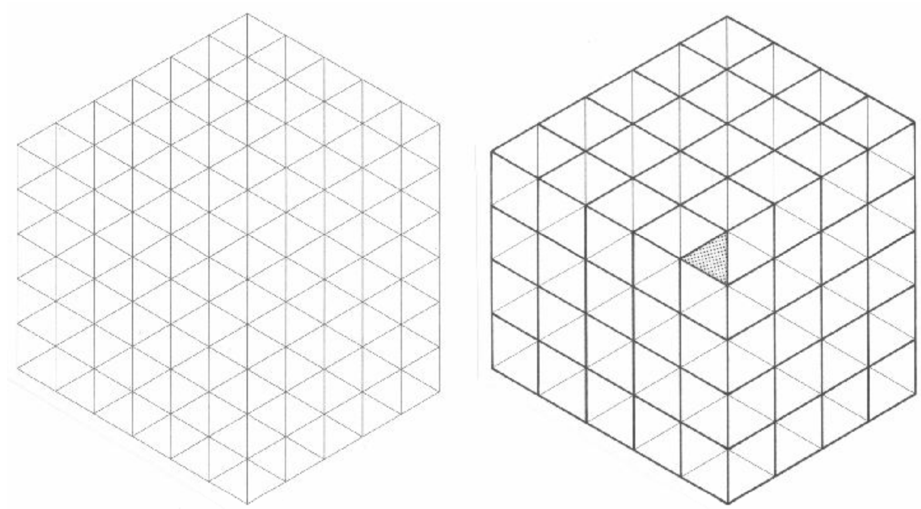


Figure 2 : *The transparent representation of an impossible 3-polycube (left) and a spiral combination of three periodic portions appearing in it (right). The shaded triangle highlights the 2-tetrahedron at the center.*

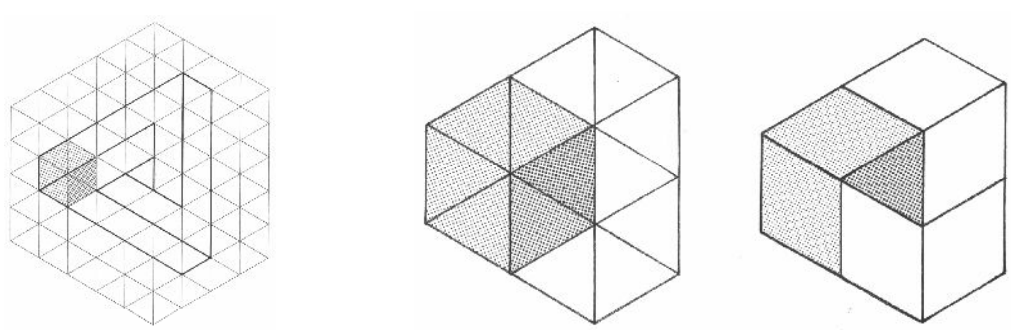


Figure 3 : *An impossible 3-polycube embeds Penrose's tribar (left end) and two scenes of the central portion derived from contracting the tribar (right two). The shaded hexagon or its part highlights a 3-cube as a base of a 4-cubic prism.*

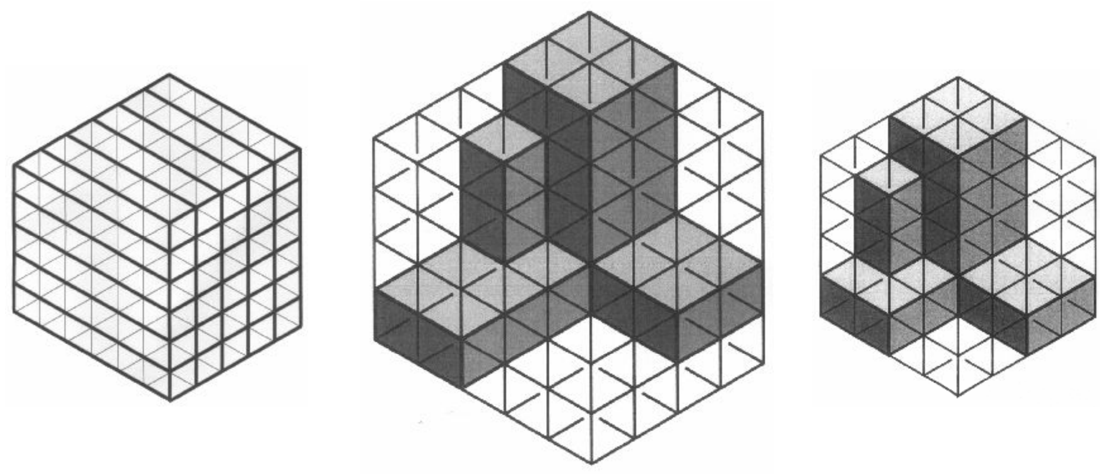


Figure 4 : *From left to right, an impossible pile of square lumbers, an impossible building embedded in an impossible polycube in a semi-regular hexagon, and a possible building in a regular hexagon. CG: P. Patrashcu.*

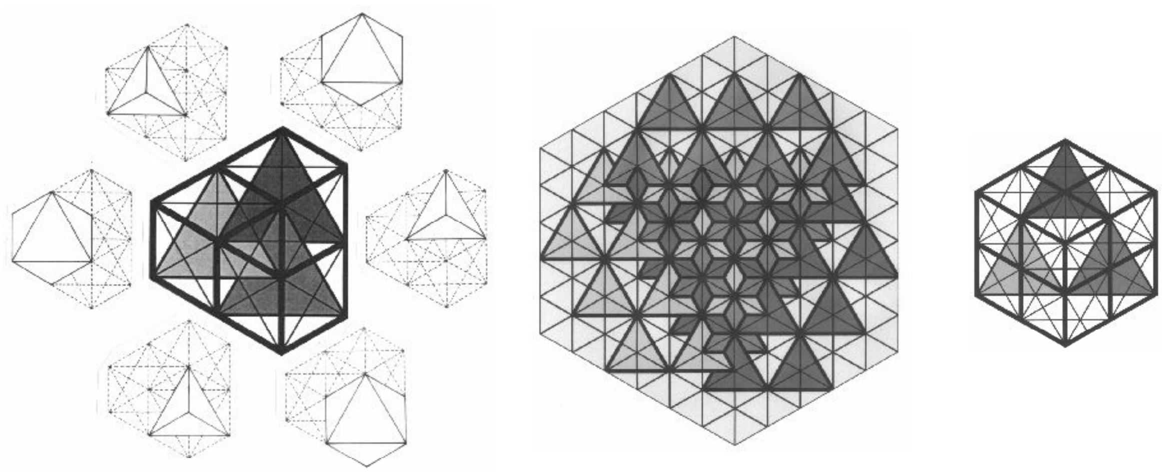


Figure 5 : *Left, the central portion of an impossible polyoctet embedded in a semi-regular hexagon. The six satellite diagrams show the impossibility; each shaded triangle must be glued to the triangle found in a parallel position in the adjoined diagram. Three of the regular triangles of them are shown in the central diagram by three shades of gray. Center, an expanded pattern of the impossible polyoctet. Right, a possible polyoctet in a regular hexagon. CG: P. Patrashcu.*

Impossible 4-polycube

An impossible 4-polycube is derived from the forcibly deformed projection into 2- or 3-space of a 4-polycube [1].

In the case on 2-space, it is composed of overlapping two kinds of rhombi including squares which are arranged in the shape of a semi-regular octagon having 4-fold symmetry (Figure 6, left). When the certain hidden lines are neglected, a spiral arrangement of four congruent periodic portions made of two kinds of rhombi including squares appear around the central square with diagonals, the projection of a 3-tetrahedron (Figure 6, right).

This pattern can embed an impossible 4-bar, Scott Kim's 4-dimensional analogue of Penrose's tribar, which is consisted of four 5-cubic prisms having 4-cubes as the bases (Figure 7, left end) [2]. In other words, the impossible 4-polycube can also be constructed by contracting an impossible 4-bar (Figure 7, right two).

In the case in 3-space, two types corresponding to two projections of a 4-cube can be considered; the rhombic dodecahedral type embedded in a 3-polycube and the oblique hexagonal prism type embedded in a polyoctet (Figure 8). According to the former, the impossible polycube shown in Figure 3 can be extended in 4-space as shown in Figure 9. On the contrary, according to the latter, a unique 4-dimensional impossible polycube may also be derived as shown in Figure 10. In this case, two kinds of rhomboids as components of the polyoctet gather around the central tetrahedron also as a component of the polyoctet.

Practical models of these can be made in 3-space, but it is impossible in 4-space.

On the other hand, the projection onto 2-space of a 4-polycube can be composed of overlapping two kinds of rhombi including squares which are arranged in the shape of a regular octagon having 8-fold symmetry (Figure 11, left). When the certain hidden lines are neglected, a radial quasi-periodic pattern composed of two kinds of rhombi, the Ammann tiling, appears around the central point (Figure 11, right) [3].

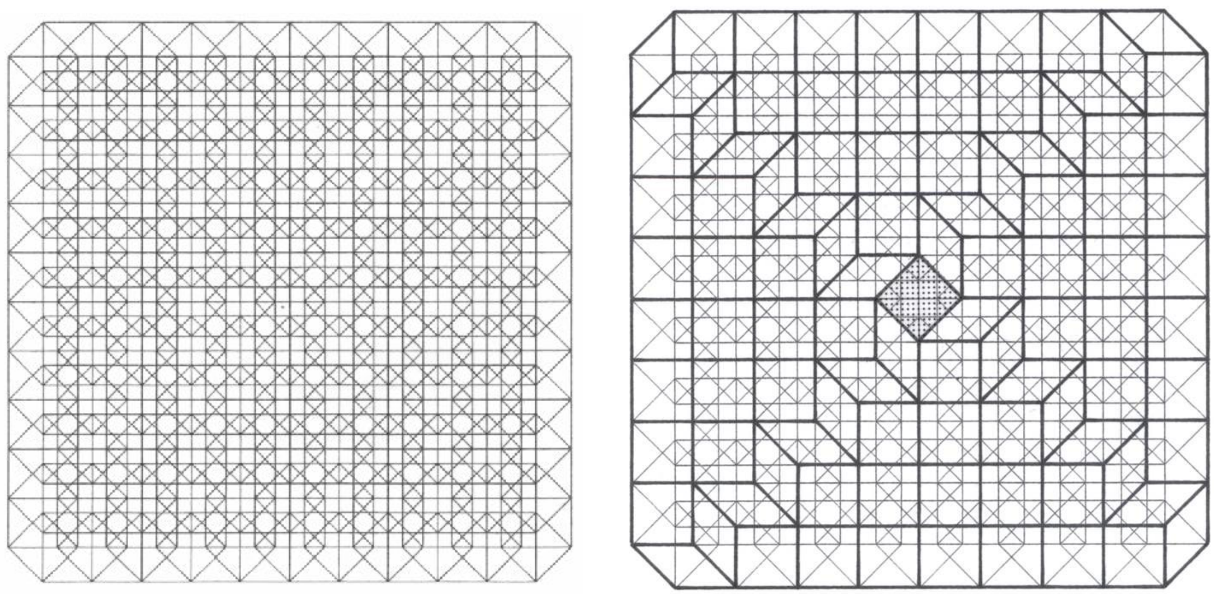


Figure 6 : *The transparent representation of an impossible 4-polycube (left) and a spiral combination of four periodic portions appearing in it (right). The shaded square highlights the projection of a 3-tetrahedron at the center. CG:M.Ishii.*

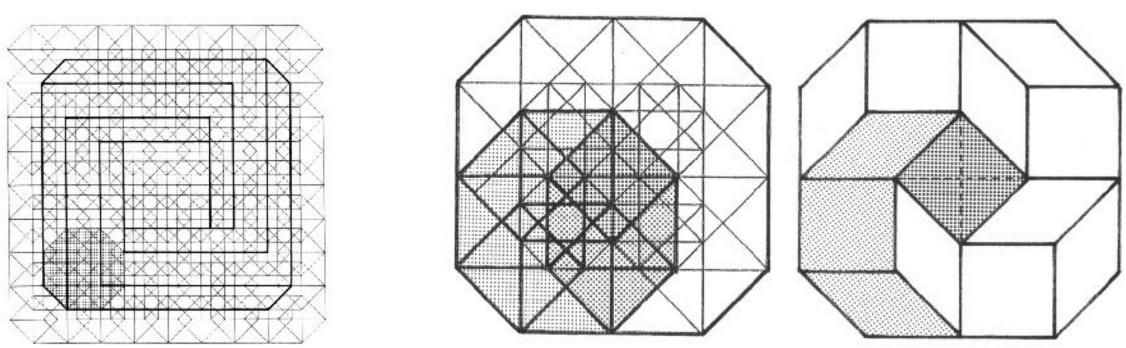


Figure 7 : *An impossible 4-polycube embeds an impossible 4-bar (left end) and two scenes of the central portion derived from contracting the 4-bar (right two). The shaded octagon or its part highlights a 4-cube as a base of a 5-cubic prism.*

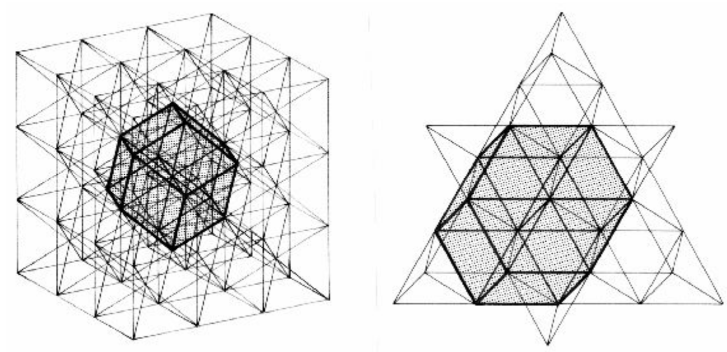


Figure 8 : *A rhombic dodecahedron in a 3-polycube (left) and an oblique hexagonal prism in a polyoctet (right).*

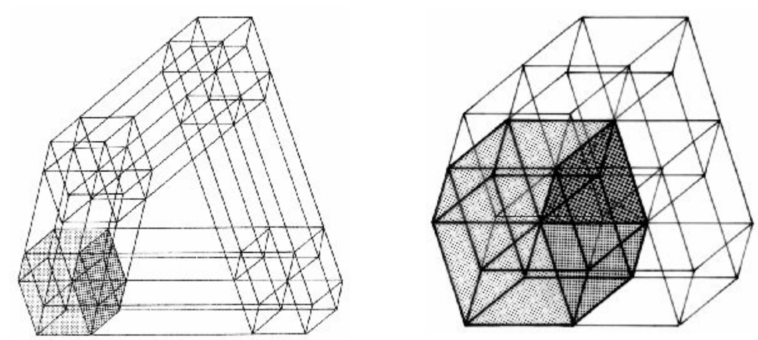


Figure 9 : *Kim's impossible 4-bar presented in 3-space (left) and the central portion of an impossible 4-polycube in 3-space derived from contracting the 4-bar (right). The shaded rhombic dodecahedron highlights a 4-cube as a base of a 5-cubic prism.*

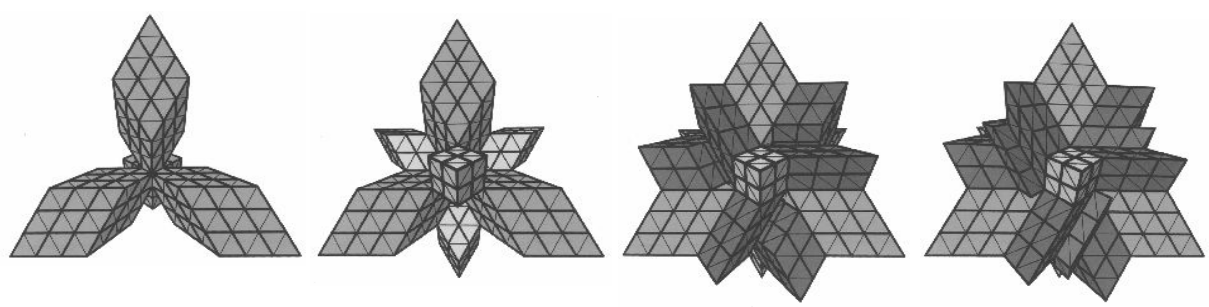


Figure 10 : *The construction process for an impossible 4-polycube embedded in a polyoctet in 3-space. All rhomboids seen in the left two figures are congruent and so are the additional rhomboids in the right two. A regular tetrahedron is hidden at the center. CG: P. Patrashcu.*

Impossible 5-polycube

An impossible 5-polycube is derived from the forcibly deformed projection into 2-, 3- or 4-space of a 5-polycube. Of these, the cases in 3- and 4-space are omitted here because they have very complicated features and hold little interest for us 3-dimensional persons.

On 2-space, an impossible 5-polycube is composed of overlapping two kinds of rhombi which are arranged in the shape of a semi-regular decagon having 5-fold symmetry (Figure 12, left). When the certain hidden lines are neglected, a spiral arrangement of five congruent periodic portions made of two kinds of rhombi appears around the central pentagon with diagonals, the projection of a 4-tetrahedron

(Figure 12, right). This pattern can embed an impossible 5-bar which consists of five 6-cubic prisms (Figure 13). In other words, an impossible 5-polycube can also be constructed by contracting an impossible 5-bar.

On the other hand, the projection onto 2-space of a 5-polycube can be composed of two kinds of rhombi which are arranged in the shape of a regular decagon having 10-fold symmetry (Figure 14, left). When the certain hidden lines are neglected, a radial quasi-periodic pattern made of two kinds of rhombi, the Penrose tiling, appears around the central point (Figure 14, right).

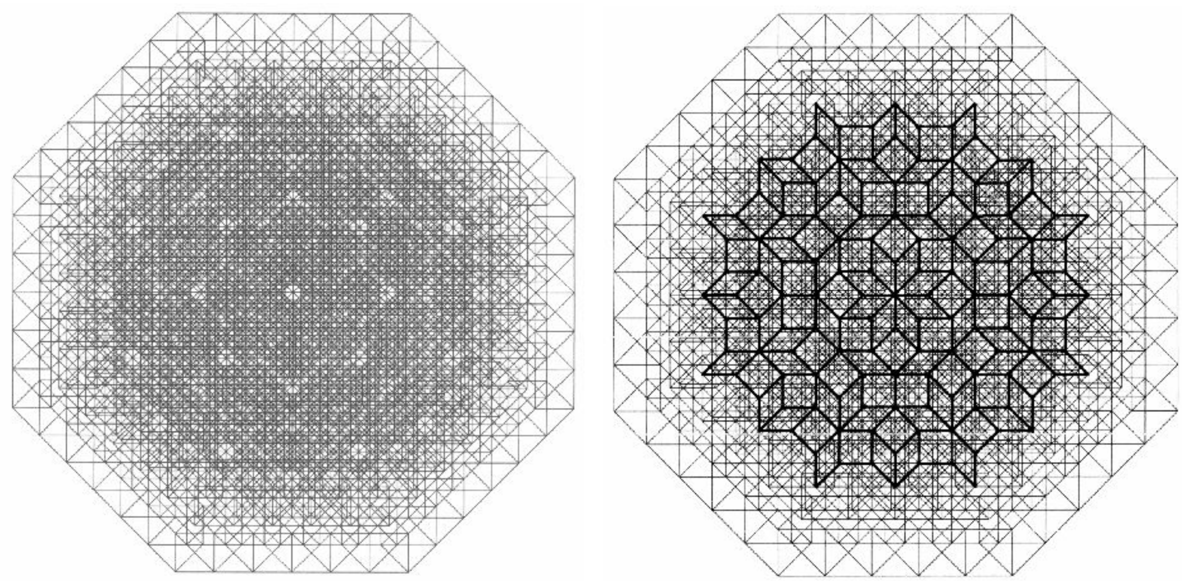


Figure 11 : *The transparent representation of the projection onto 2-space of a 4-polycube (left) and a radial quasi-periodic pattern appearing in it (right). CG: M. Ishii.*

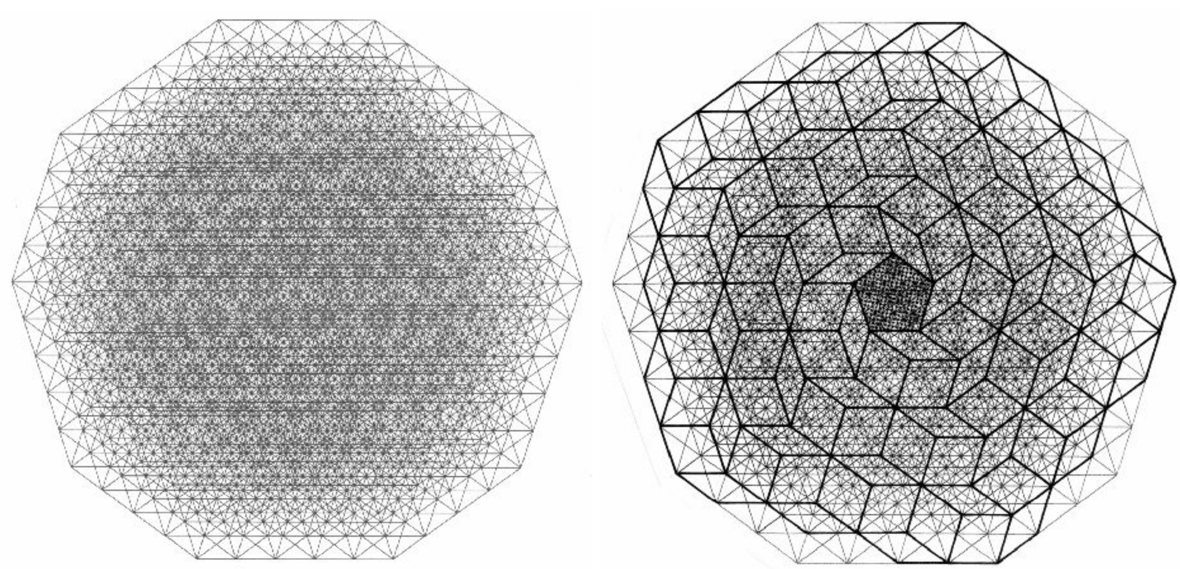


Figure 12 : *The transparent representation of an impossible 5-polycube (left) and a spiral combination of five periodic portions appearing in it (right). The shaded pentagon highlights the projection of a 4-tetrahedron at the center. CG: M. Ishii.*

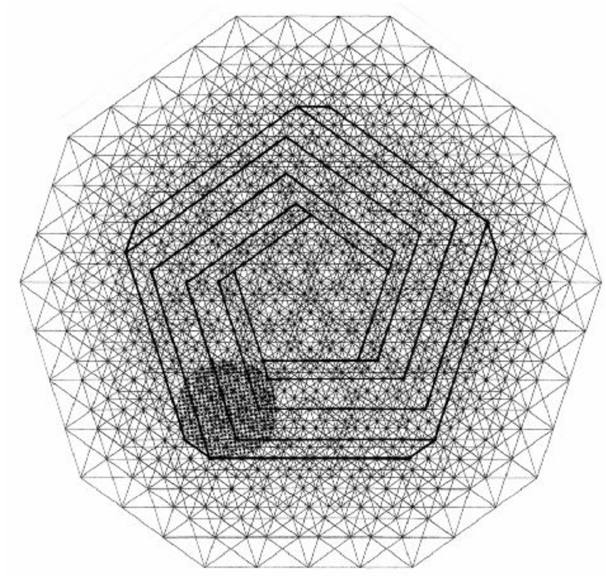
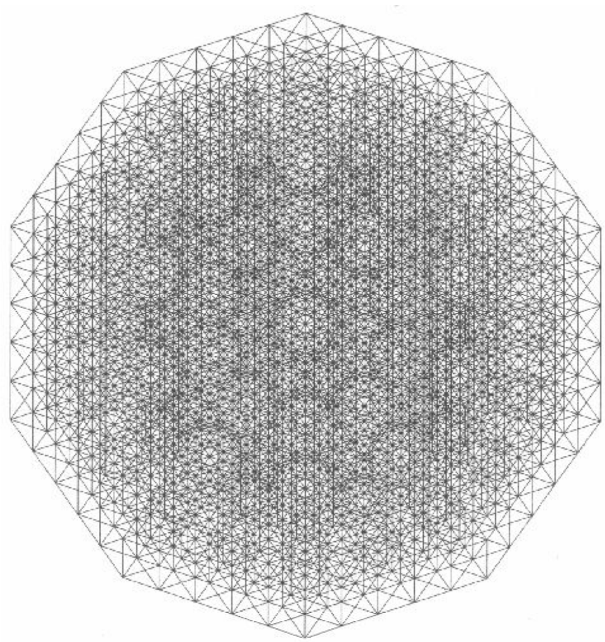


Figure 13 : *An impossible 5-bar embedded in an impossible 5-polycube. The shaded decagon highlights a 5-cube as a base of a 6-cubic prism. CG: M. Ishii.*



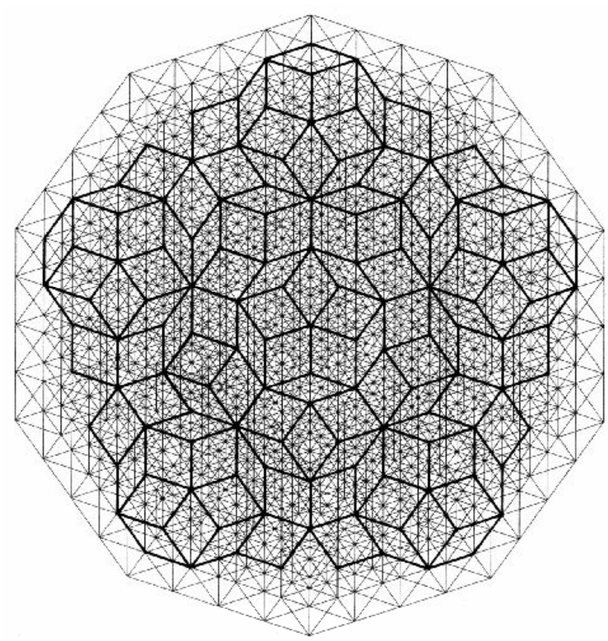


Figure 14 : *The transparent representation of the projection onto 2-space of a 5-polycube (left) and a radial quasi-periodic pattern appearing in it (right). CG: M. Ishii.*

Conclusions

In general, we can obtain an impossible n-polycube by forcibly deforming the projection into (n-1)- or lower-space of an n-polycube. Of these, only the case represented on 2-space will attract attention of 3-dimensonal persons for the simple features.

The impossible n-polycube is composed of overlapping [n/2] kinds of rhombi which are arranged in the shape of a semi-regular 2n-gon having n-fold symmetry. When certain hidden lines are neglected, a spiral arrangement of *n* congruent periodic portions made of [n/2] kinds of rhombi appears around the central regular n-gon with diagonals, the projection of a (n-1)-tetrahedron. It can embed an impossible n-bar which consists of *n* numbers of (n+1)-cubic prisms. In other words, the impossible n-polycube can also be constructed by contracting an impossible n-bar. On the other hand, the projection onto 2-space of an n-polycube is composed of overlapping [n/2] kinds of rhombi which are arranged in the shape of a regular 2n-gon having 2n-fold symmetry. When the certain hidden lines are neglected, a radial quasi-periodic pattern composed of [n/2] kinds of rhombi appears around the central point.

Figure 15 shows the 6-dimensional case as a general example in even-numbered dimensional space. The already mentioned 5-dimensional case is that in odd-numbered dimensional space.

While these objects are nothing more but artistic amusements, they show projections into 2- or 3-space of higher-dimensional figures. From these projections, it is possible to get abundant geometric knowledge about higher-spaces.

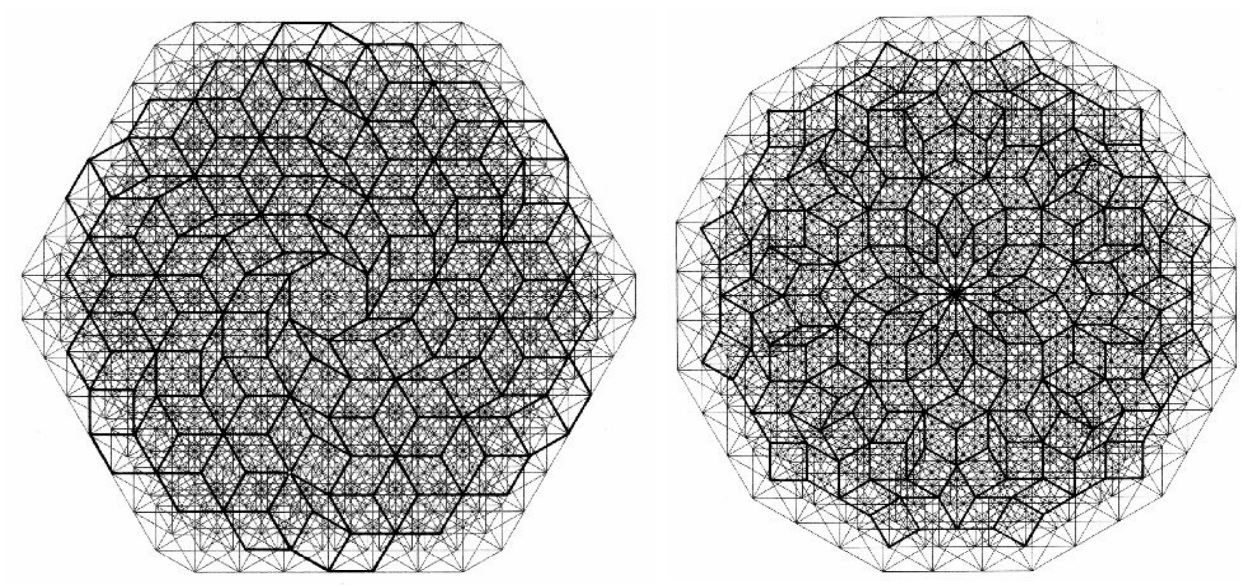


Figure 15 : *Left, an impossible 6-polycube embeds a spiral combination of six periodic portions made of three kinds of rhombi around a hexagon. Right, the projection of a 6-polycube embeds a radial quasi-periodic pattern made of the same three kinds of rhombi around the central point. CG: M. Ishii.*

References

[1] K. Miyazaki, "Impossible Polycube – four-dimensional version", *Experience-centered Approach and Visuality In The Education of Mathematics and Physics*, S. Jablan, et al. ed., the Kaposvar University (2012), pp.180-182.

[2] S. Kim, "An Impossible Four-Dimensional Illusion", *Hypergraphics: Visualizing Complex Relationships in Art, Science and Technology,* D. Brisson ed., Westview Press (1978), pp.187-239.

[3] B.Grünbaum, G.C.Shephard, "Tilings and Patterns", W.H.Freeman (1987), pp.556-557.

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