A Fun Approach to Teaching Geometry and Inspiring Creativity

Ioana Browne, Michael Browne, Mircea Draghicescu*, Cristina Draghicescu, Carmen Ionescu
ITSPHUN LLC
2023 NW Lovejoy St.
Portland, OR 97209, USA
itsphun@itsphun.com

Abstract

ITSPHUN™ geometric shapes can be easily combined to create an infinite number of structures both decorative and functional. We introduce this system of shapes, and discuss its use for teaching mathematics. At the workshop, we will provide a large number of pieces of various shapes, demonstrate their use, and give all participants the opportunity to build their own creations at the intersection of art and mathematics.

Introduction

Connecting pieces of paper by sliding them together along slits or cuts is a well-known method of building 3D objects ([2], [3], [4]). Based on this idea we developed a system of shapes[1] that can be used to build arbitrarily complex structures. Among the geometric objects that can be built are Platonic, Archimedean, and Johnson solids, prisms and antiprisms, many nonconvex polyhedra, several space-filling tessellations, etc. Guided by an user's imagination and creativity, simple objects can then be connected to form more complex constructs.

ITSPHUN pieces made of various materials[2] can be used for creative play, for making functional and decorative objects such as jewelry, lamps, furniture, and, in classroom, for teaching mathematical concepts.[3]

[1]Patent pending.
[2]We used many kinds of paper, cardboard, foam, several kinds of plastic, wood and plywood, wood veneer, fabric, and felt.
[3]In addition to models of geometric objects, **ITSPHUN** shapes have been used to make birds, animals, flowers, trees, houses, hats, glasses, and much more - see a few examples in the photo gallery at www.itsphun.com.

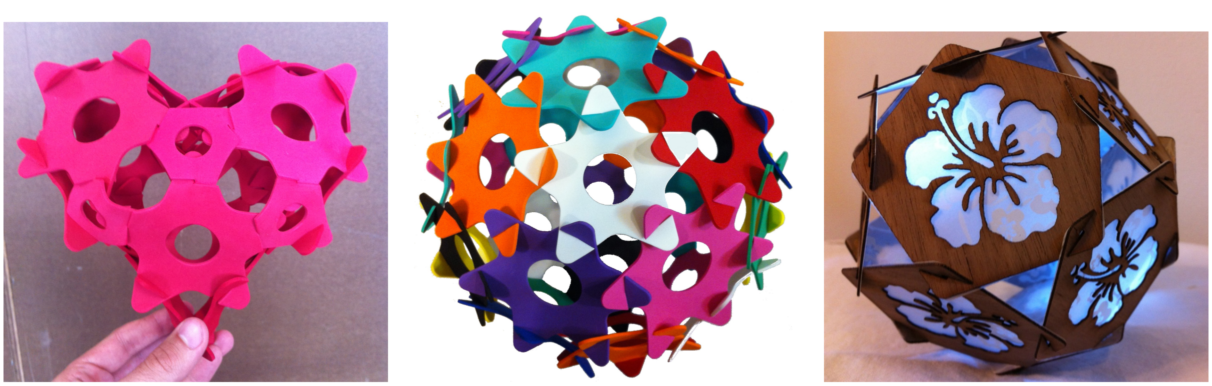

Figure 1 : *Heart, truncated icosahedron (buckyball), and dodecahedron*

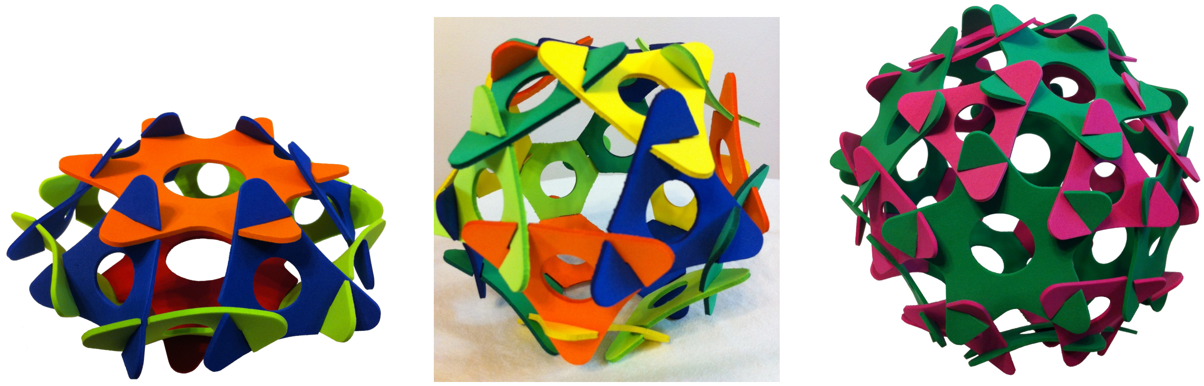

Figure 2 : *6-sided antiprism, icosahedron, icosidodecahedron*

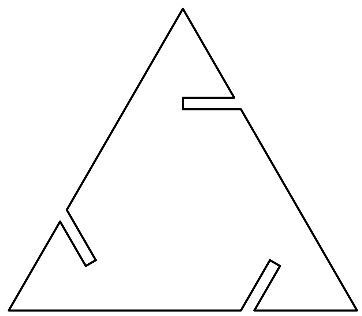

Figure 3 : *A piece with 3 notches*

Brief History

ITSPHUN, which stands for Interlocking Triangles, Squares, Pentagons, and Hexagons Using Notches,[4] started when we took a number of laser-cut EVA foam shapes to the 2012 Bay Area Maker Faire. The presentation proved to be extremely popular with both children and adults (see some pictures at `www.itsphun.com`). This led to an exhibit at The Tech Museum in San Jose, more fairs, and eventually to a startup company, ITSPHUN LLC.

ITSPHUN was awarded a Maker Faire Editor's Choice Blue Ribbon at the 2012 World Maker Faire New York and was selected by Scientific American for its 2012 educational gift guide [1].

Geometry

Intuitively, an **ITSPHUN** piece is a plane figure with a number of congruent, roughly rectangular, cutouts or *notches* (Figure 3). The width of a notch is determined by the thickness of the fabrication material; to simplify the presentation we will assume here that the width is 0, which will work for (thin) paper pieces.

The n notches of a piece are symmetrically placed around the center of an imaginary regular n-sided *underlying polygon*. More precisely, the notches (represented by thick lines in Figure 4) are placed along the sides of the (shaded) underlying polygon, extending in the same direction (here clockwise) from the midpoint of the corresponding side.

In Figure 4 the underlying polygons form part of the outside contour of the pieces, but in general, once

[4] Shapes can be based on any regular polygon, even if it does not appear in our name, e.g., octagon, decagon, etc.

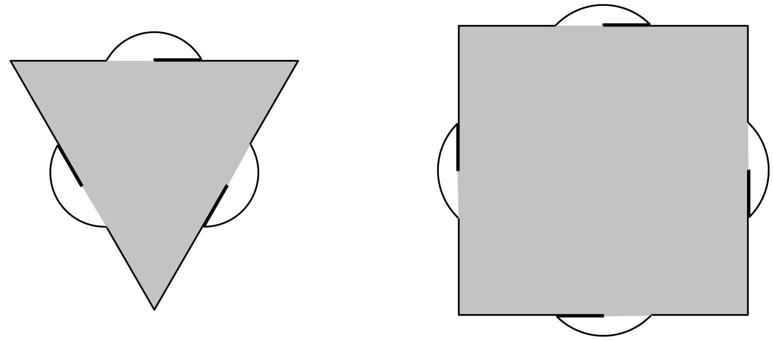

Figure 4 : *Notch placement wrt the underlying regular polygon*

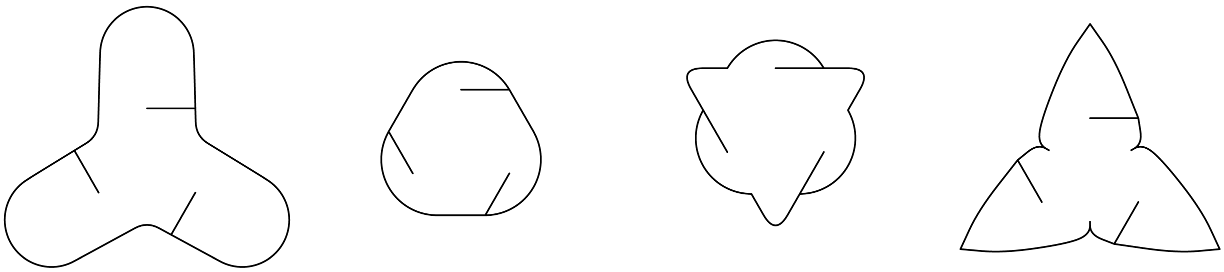

Figure 5 : *Various pieces with the same notch configuration*

the notches are placed, the underlying polygon plays no role in the overall shape of the piece. In fact, the outer contour is not important for connectivity, does not even have to have the rotational symmetry of the notches and can be chosen based on aesthetic considerations.[5] Figure 5 shows several pieces with exactly the same notch configurations. The only restriction that we impose is that the notch length is at most half the length of the side of the underlying polygon (see Figure 4), even when the underlying polygon is not apparent in the piece contour. This restriction avoids self-intersections when building a convex 3D object.[6]

Four Figure 4 triangles can be connected to form a regular tetrahedron by sliding them together along their notches.[7] More general, any pieces with the same number of notches can be connected to form a 3D

[5]Of course, when cutting real shapes we must leave enough space around the notches - see the rounded "bulges" in Figure 4.

[6]Self-intersections are still possible when trying to build certain nonconvex structures.

[7]More precisely, the underlying triangles of the pieces form a tetrahedron. When no confusion is possible we will call a piece

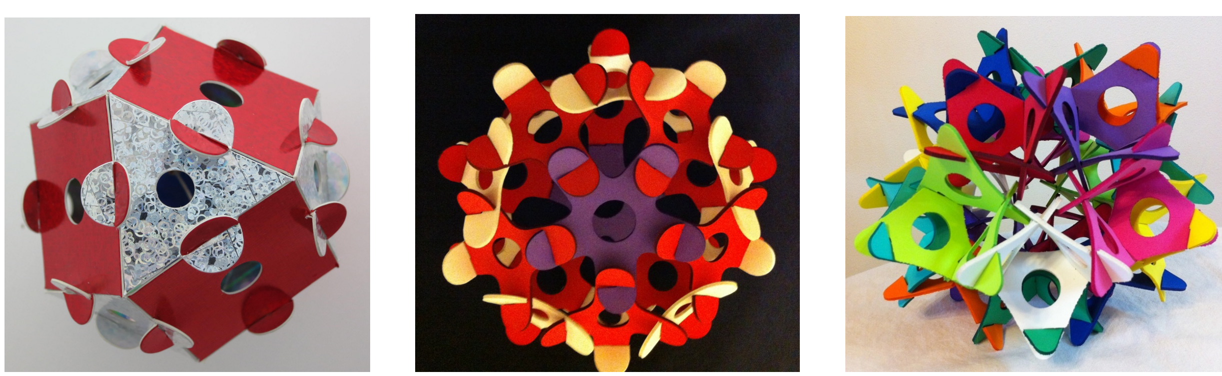

Figure 6 : *Cuboctahedron, pentagonal gyrobicupola, great dodecahedron*

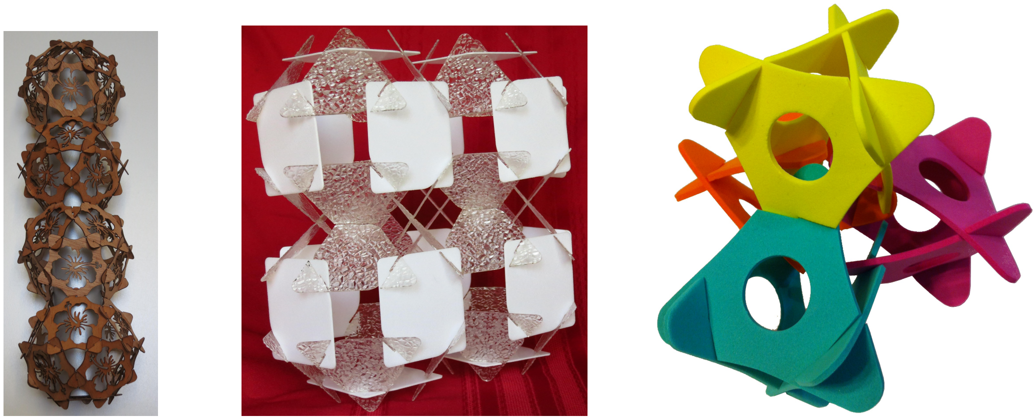

Figure 7: *Objects obtained by connecting simpler polyhedra (dodecahedra, cuboctahedra, and tetrahedra, respectively)*

object based on a Platonic solid iff (a) their notches have the same length, and (b) the underlying polygons are congruent (in other words, the notches are placed at the same distance from the rotation center). For example, any combination of 4 pieces from Figure 5 can be connected into a tetrahedron.

Pieces with a different number of notches can be connected to form a semi-regular polyhedron iff (a) their notch length is the same, and (b) the sides of the underlying polygons have the same length. For example 8 Figure 4 triangles can be connected with 6 Figure 4 squares to form a cuboctahedron as in Figure 6. In general, the parts of the pieces (here the "bulges" around the notches) that are outside the underlying polygon will "stick out" in the final construct, as in Figure 6. When working with pliable materials such as paper, it is possible to "reverse" the connections so that these parts are hidden inside the object,[8] in which case the constructed object will resemble a true cuboctahedron.

A coherent system of pieces with $3, 4, 5, \ldots$ notches can thus be obtained as follows:

1. Choose a common length s of the sides of all underlying polygons.
2. Choose a notch length $l \leq s/2$. In general, a longer notch results in stronger connections but limits the possibilities for the overall shape of a piece and makes it more difficult to assemble complex structures; depending on the manufacturing material, there is a limit on how short a notch can be.
3. Place the notches as outlined above.
4. Design the overall contour for each piece around the placed notches.

Connecting Polyhedra

Two polyhedra can be joined together by lining them up on two congruent faces, removing both faces (one from each polyhedron), and connecting the newly freed notches. This process can be repeated resulting in arbitrarily complex objects (see Figure 7). For example, to obtain the last object in Figure 7 we could start with a tetrahedron T and append a tetrahedron onto each face of T. In the process the faces of T (and a face from each of the four added tetrahedra) are removed, so the new object has 12 triangular faces. Of course, in practice we will not start with T but directly connect the 12 triangles together into the final object. [9]

with three notches a "triangle", the 3D construct obtained by connecting 4 such triangles a "tetrahedron", etc.

[8]In other words, the "stitches" can be inside rather than outside.

[9]This joining method has a mechanical limitation: we cannot join coplanar faces except when working with pliable materials. This problem arises when the two dihedral angles of the polyhedra being joined together are supplementary. In practice this does

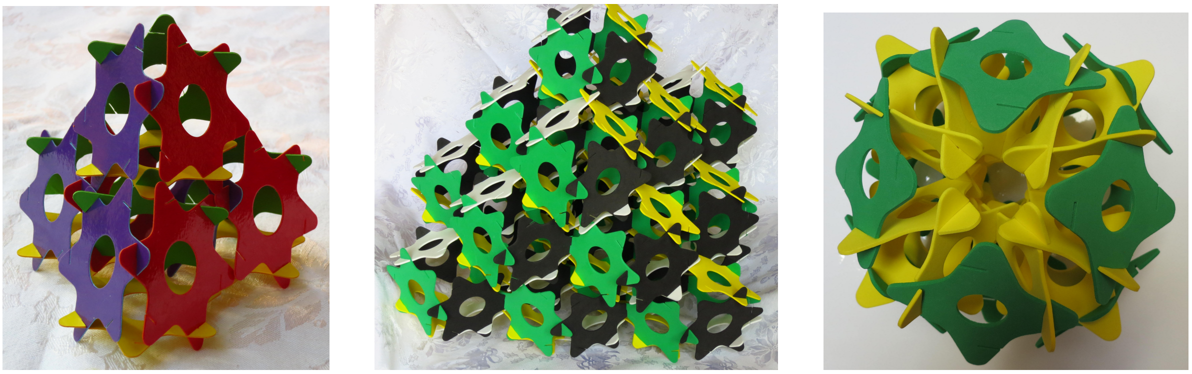

Figure 8: *Space-filling tessellations or honeycombs: quarter cubic (or bitruncated alternated cubic), bitruncated cubic, gyrated triangular prismatic*

The **ITSPHUN** pieces of an object build by joining simpler polyhedra, as described above, correspond to the *external* (outside) faces of the final geometric object (there are no pieces inside the object). We can often build objects that are both more decorative and structurally solid by removing some of these external faces and using the freed notches to add some *internal* ones, while still preserving the property that only two faces meet at an edge. The end result is a 3-D surface consisting of some of the faces of the original polyhedra (before joining them together), some internal and some external. Since **ITSPHUN** pieces are not, in general, complete polygons (see, for example the circular holes) these constructs combine the features of a solid and a wire frame model of a complex geometric object, revealing both its outside shape and its internal structure.

This technique is illustrated by the models of some space tessellations in Figure 8. The first object, built out of 16 hexagons of 4 colors, is a fragment of the quarter cubic (or bitruncated alternated cubic) honeycomb that fills the space with truncated tetrahedra and tetrahedra in the $1:1$ ratio. In our version, the tetrahedra are missing completely and the truncated tetrahedra have only their hexagonal faces. The notches that are freed by removing the triangular faces are used to connect the truncated tetrahedra together. The middle object in Figure 8 is a fragment of the bitruncated cubic honeycomb, a space-filling of truncated octahedra cells. Here the square faces of the truncated octahedra are missing thus freeing enough notches for connecting the cells together. The resulting structure has all the hexagonal faces, including the internal ones. Finally, in the gyrated triangular prismatic honeycomb fragment in Figure 8 the triangular faces of the triangular prism cells are missing but all the square faces are present. Note also that all the objects in Figure 8 have free notches that allow the constructions to continue *ad infinitum*.

Classroom activities: teaching mathematics through play

One of the most satisfying moments when presenting **ITSPHUN** is watching young (and sometimes not so young) participants having an "aha!" moment when they discover the symmetry pattern of a mathematical object and the joy that follows when they are able to make a copy of the object by following the just discovered rule. Many become hooked, ask for more and more complex objects to build and then start inventing their own. In a classroom setting, a simple activity in which the teacher presents an object as a model and asks the students to reproduce it requires minimal teacher intervention and helps develop essential skills such as pattern recognition and spatial vision. In a more advanced activity the teacher can describe an object without showing a model, for example: make a cube, then replace each face by a square pyramid.

not happen except when joining cubes, and building with cubes can be done better with many other construction toys.

A teacher can build on the initial interest and introduce some abstract concepts, using the fact that being able to see, manipulate, and make 3D objects greatly facilitates the understanding. Here are some simple questions that can be asked in a geometry class during a unit on volume and surface area of solids (here V = volume and A = surface area):

- Make a pentagonal right prism; how can we compute its V and A?
- Make it twice as high; how do V and A change?
- Make a cube, compute its V and A; now double its side - how do V and A change?
- How many triangular prisms can we fit into a hexagonal one and what does this say about their V?
- Measure the side of a cube in inches and compute its V, then repeat this in centimeters. Can you tell what V would be in feet without doing any additional measurements?
- Make a right pentagonal prism by stacking pentagonal pieces; what can we say about its V? Now shift the pieces to obtain an oblique prism - did V change? (Cavalieri's principle)

Here are some topics related to the objects that can be constructed with **ITSPHUN** that can be presented in a math club:

- Platonic solids: history, duality, proof of Euler's formula, proof that there are exactly 5 Platonic solids.
- Other convex uniform polyhedra: prisms and antiprisms, Archimedean solids: symmetry, operations (truncation, rectification, expansion, etc.)
- Johnson solids: types (pyramids, cupolae, rotunda), operations (elongation, gyration, etc.)
- Nonconvex polyhedra, stellations.
- Infinite structures, plane and space filling.
- Beyond 3D: polytopes. As an example, the connected tetrahedra in Figure 7 can be seen as a net of the *5-cell* or *pentachoron*, the four-dimensional analogue of the tetrahedron.

In conclusion, we want to reiterate that just playing with **ITSPHUN** can help develop an intuition for some fundamental mathematical ideas. In particular, the concept of *symmetry* comes up again and again and even very young children can intuitively detect "more" and "less" symmetrical constructs. In addition to the symmetry of the objects themselves, the use of color is pervasive and choosing an aesthetically pleasing coloring scheme when building an object adds a new dimension to any activity. Questions such as why one object is "nicer" than another one, what coloring schemes would "fit" a certain object and why one coloring is "better" than another one can be used to introduce older students to geometrical transformations and basic group theory and give an artistic justification for studying these abstract subjects.

References

[1] All I Want for Christmas Is Dinosaur Dung, Scientific American's educational gift guide for 2012. *Scientific American*, pages 18–19, December 2012.

[2] George W. Hart. "Slide-Together" Geometric Paper Constructions. In *Bridges for Teachers, Teachers for Bridges, Workshop Book*, pages 31–42, 2004.

[3] George W. Hart. Modular Kirigami. In *Proceedings of Bridges Donostia, San Sebastian, Spain*, pages 1–8, 2007.

[4] Jace Miller and Ergun Akleman. Edge-Based Intersected Polyhedral Paper Sculptures Constructed by Interlocking Slitted Planar Pieces. In *Proceedings of Bridges Conference on Mathematical Connections in Art, Music, and Science, Leeuwarden, The Netherlands*, July 2008.