Proceedings of Bridges 2013: Mathematics, Music, Art, Architecture, Culture

Visualizing 3-Dimensional Manifolds

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Abstract

Given a parametrized 3-dimensional manifold sitting in 4-dimensional space, we wish to visualize it by looking at its intersections with 3-dimensional hyperplanes. The intersections are 2-dimensional surfaces in 4-space which can then be projected into 3-space for visualization. In this paper I present an algorithm for displaying these surfaces of intersection using computer plotting applications (e.g. *Mathematica*, MATLAB, etc.).

Methodology

Let $\varphi : U \to \mathcal{M} \subset \mathbb{R}^4$ be a parametrization of a 3-manifold $\mathcal{M}$ where $U \subset \mathbb{R}^3$ is a region in the parameter space of φ. Let $f : \mathbb{R}^4 \to \mathbb{R}$ be a smooth function whose differential Df never vanishes, so that the level sets $f^{-1}(c)$ are 3-dimensional hypersurfaces in $\mathbb{R}^4$. For this paper, f is taken to be the dot product $f(\vec{v}) = \vec{\eta} \cdot \vec{v}$ for some nonzero vector $\vec{\eta} \in \mathbb{R}^4$; the level sets $f^{-1}(c)$ are the hyperplanes in $\mathbb{R}^4$ perpendicular to $\vec{\eta}$. Let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ be some projection or mapping, for this paper π is taken to be the projection onto the xyz-hyperplane given by $\pi(x, y, z, w) = (x, y, z)$.

Let $c \in \mathbb{R}$ be some value and consider the hypersurface $f^{-1}(c) \subset \mathbb{R}^4$. We wish to compute the intersection $\mathcal{M}' = (\mathcal{M} \cap f^{-1}(c))$ and then project this surface from the ambient space $\mathbb{R}^4$ to $\mathbb{R}^3$ via the map π. The slice $\mathcal{M}'$ is in general a 2-dimensional submanifold of $\mathcal{M}$, and its projected image $\pi(\mathcal{M}') = \pi(\mathcal{M} \cap f^{-1}(c)) \subset \mathbb{R}^3$ is what we wish to observe as a 2-dimensional manifold in 3-space.

Note that if $\mathcal{M}$ is indeed parametrized by the patch $\varphi : U \to \mathcal{M}$, in particular if φ is onto, then every point $m \in \mathcal{M}$ has a preimage in the set U, so $\varphi(U)$ and $\mathcal{M}$ are equal as sets. We may take the slice $\mathcal{M}' \subset \mathcal{M}$ and consider its preimage under φ as a subset $U' \subset U$ in the parameter sapce of φ. Let U' be defined in this way; then $U' := \varphi^{-1}(\mathcal{M}') = \varphi^{-1}(f^{-1}(c) \cap \mathcal{M}) = (f \circ \varphi)^{-1}(c) \subset U \subset \mathbb{R}^3$.

The original intersection $\mathcal{M}'$ can be recovered by mapping the set $U' \subset \mathbb{R}^3$ back into $\mathbb{R}^4$ via φ, since by definition we have $\varphi(U') = \varphi(\varphi^{-1}(\mathcal{M}')) = \mathcal{M}'$. Mathematically, this fact is a trivial consequence of the stipulation that φ is onto. Computationally, however, this is important because $U' = (f \circ \varphi)^{-1}(c)$ can be computed directly as a level set (or "isosurface") in the parameter space U inside $\mathbb{R}^3$. Once U' is computed, $\mathcal{M}' = \varphi(U')$ is recovered by mapping U' back into $\mathbb{R}^4$ via φ; from there we project $\mathcal{M}'$ down to $\mathbb{R}^3$ via π. The projected image of the slice $\mathcal{M}'$ is the set $\pi(\mathcal{M}') = \pi(\varphi(U')) = (\pi \circ \varphi)(U') = (\pi \circ \varphi)\big((f \circ \varphi)^{-1}(c)\big)$. Thus, $\pi(\mathcal{M}')$ is computed as the image of the isosurface $(f \circ \varphi)^{-1}(c) \subset \mathbb{R}^3$ under the map $(\pi \circ \varphi) : \mathbb{R}^3 \to \mathbb{R}^3$.

$$
\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "\varphi"] & \mathbb{R}^4 \arrow[r, "f"] \arrow[d, "\pi"] & \mathbb{R} \\
& \mathbb{R}^3 &
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "\varphi"] & \mathcal{M} \arrow[r, "f"] \arrow[d, "\pi"] & \mathbb{R} \\
& \mathbb{R}^3 &
\end{tikzcd}
\qquad
\begin{tikzcd}
U' \arrow[r, "\varphi"] \arrow[rr, dashed, bend left, "(f \circ \varphi)"] \arrow[dr, dashed, "(\pi \circ \varphi)"'] & \mathcal{M}' \arrow[r, "f"] \arrow[d, "\pi"] & c \\
& \pi(\mathcal{M}') &
\end{tikzcd}
$$

$$\pi(\mathcal{M}') = (\pi \circ \varphi)\big((f \circ \varphi)^{-1}(c)\big)$$

Examples and Connections to Art

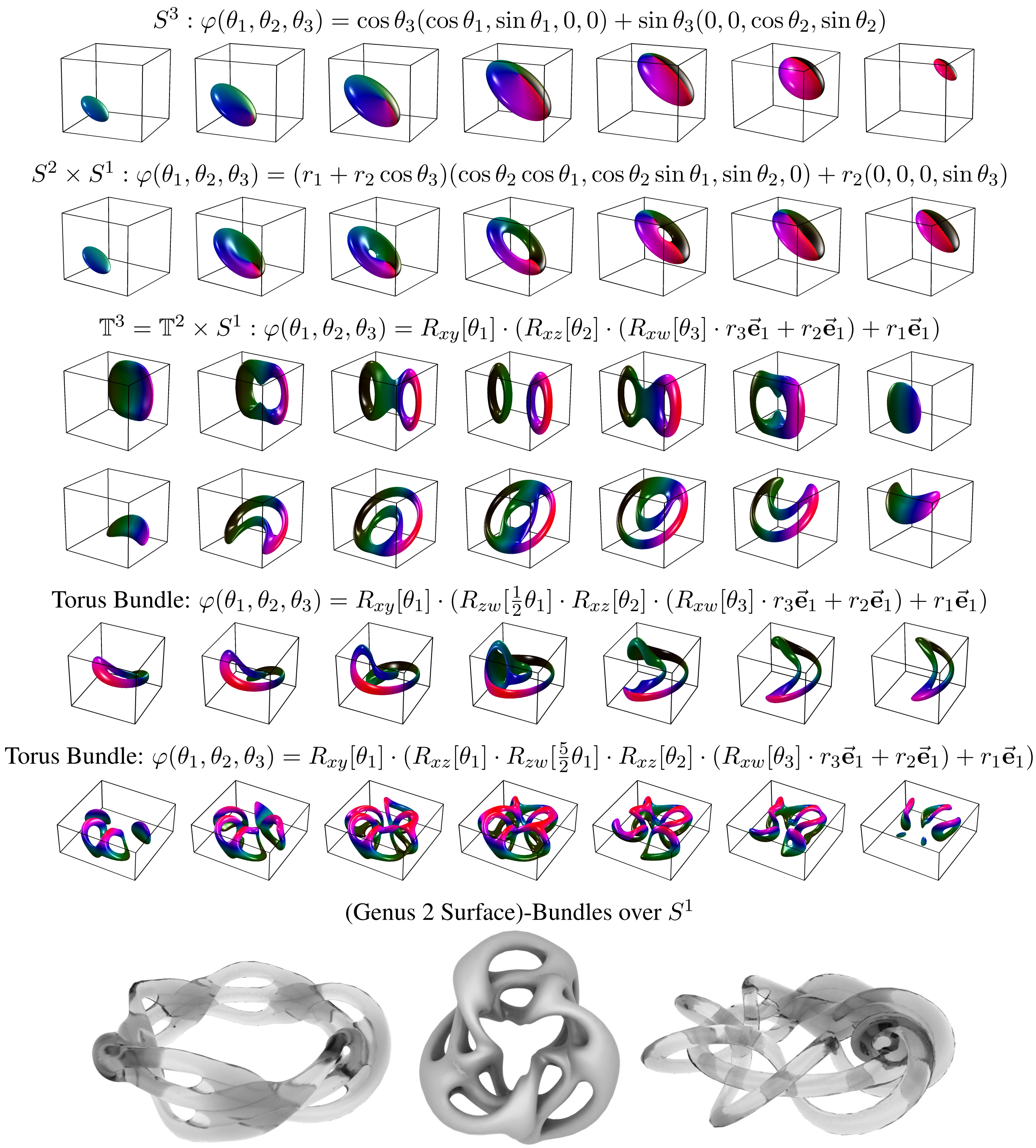

Figure 1: *Slices of the manifolds S^3, $S^2 \times S^1$, $\mathbb{T}^3$, and two different $\mathbb{T}^2$-bundles over S^1. In the above parametrizations, $R_{ab}[\theta]$ denotes the rotation in the ab-plane by angle θ, and $\vec{\mathbf{e}}_1 = (1, 0, 0, 0)$ denotes the x-coordinate vector. Also shown are the $\{w{=}0\}$ slices of various 3-manifolds parametrized as fiber bundles over S^1 whose fibers are (oriented) genus-2 surfaces which perform any number of twists about two rotational planes as they trace around the base S^1. These are all examples of aesthetically pleasing and geometrically interesting shapes that can be generated efficiently as slices of 3-dimensional manifolds.*