One Mucuboctahedron: Four Ways to View It

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Abstract

Our first attempt at building the mucuboctahedron naturally resulted in a symmetric tetrahedral structure. As we explored other places to cut the infinite Archimedean polyhedron, it revealed a cubic lattice structure, a ring around a hexagonal hole structure, and an octahedral structure. This inspired some further research into its geometric properties.

Introduction

A discussion of infinite polyhedra in *The Symmetries of Things* [2] by Conway, Burgiel and Goodman-Strauss inspired the first author to explore infinite Archimedean polyhedra. The first exploration was an embroidered fabric model of the murhombicuboctahedron (Figure 1, left). The second in this series is an embroidered felt version of the mucuboctahedron (Figure 1, right). The vertex configuration for this polygon is (6.4.6.4), that is, the polygons about a vertex are hexagon, square, hexagon, square, in that order.

Figure 1 : *Murhombicuboctahedron (left) and mucuboctahedron*

The murhombicuboctahedron (6.4.4.4) seemed a rather straightforward model with its chambers and tunnels arranged in a tidy cubical array. The mucuboctahedron on the other hand features paired chambers of truncated octahedra with four hexagonal faces removed. The two chambers have hexagonal faces colored blue, yellow, purple and red in the same order on the outside but are oriented at a 180 degree rotation from each other. This has the effect of staggering the chambers from one pair to the next in three directions.

The mucuboctahedron

The description of this solid provided by Conway et al ([2], p. 336) is:

> The multiplied cuboctahedron (CO): 6.4.6.4 Group 2+:2. Hyp *642. The faces are the squares of our truncated octahedral together with half their hexagons, ...

The reason why it is called a multiplied cuboctahedron rather than a multiplied truncated octahedron is a story in itself, into which we shall not go.

Depending upon where the infinite polyhedron is trimmed the mucuboctahedron shows several different patterns. In Figure 1 we saw the tetrahedral symmetry and structure highlighted. In Figure 2 below we see a cubic structure that reflects its basic lattice structure (left), an octahedron (center) and finally a ring of cells that moves about a central hexagon (right). So we have one infinite polyhedron that presents itself in at least four different ways depending upon how we look at it.

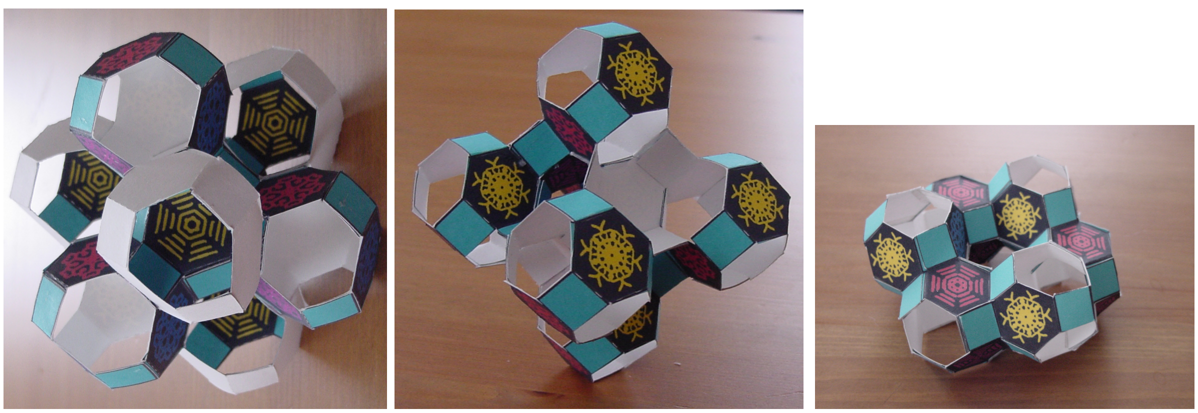

Figure 2 : *Cubic, Octahedral and Hexagonal structures*

Some historical notes

H. S. M. Coxeter and J. F. Petrie first explored infinite polyhedra as 16 year olds and later wrote about the Coxeter-Petrie polyhedra in a paper "Regular Skew Polyhedra in Three and Four Dimensions, and Their Topological Analogues" [3] which was reprinted in 1968 as part of *Twelve Geometric Essays* [4], and reprinted by Dover books in 1999 [5]. Their focus was entirely on the regular infinite polyhedra. Conway et al observe, "We believe that nobody has yet enumerated the hundreds of 'Archimedean' polyhedra in 3-space" ([2], p. 340).

In 1974, architects Wachman, Burt and Kleinmann wrote a paper that included an extensive list of infinite polyhedra. Michael Burt has recently revisited this and maintains copies of these resources on his web site [11]. In 1977, A. F. Wells wrote a book that included an extensive table of uniform honeycombs and infinite polyhedra [12].

A number of people are doing current research in this area. Doug Dunham explores hyperbolic tessellations printed on infinite polyhedra [6]. Vladimir Bulatov has made extensive explorations of hyperbolic space and its visualization [1]. Egon Schulte has written extensively about chiral polyhedra in ordinary space [10]. He has also written papers about symmetries of polytopes and polyhedra. These are also available on his web site. Steve Dutch [7] has explored virtual representations of hyperbolic tessellations on skew polyhedra on his web site [7]. A website maintained by Melinda Green [8] provides some discussion of infinite regular polyhedra and a table of them.

Examining the model

The fabric model (Figure 1, right) naturally displays the tetrahedral structure with its three and four way rotational symmetries in several directions as well as multiple mirror symmetries through planes that cut faces into congruent polygons. We know that the tetrahedron does not tile space; however, the truncated octahedra do. In this case the tiling takes place in the form of paired chambers that are inverted with respect to each other and rotated 180 degrees. We also see the face-centered cubic lattice structure generated by three vectors emanating from one vertex of the tetrahedral model and extending half-way to the other three vertices.

Trimmed to the cubic presentation (Figure 2, left) where a normal cubic lattice is demonstrated, we see that the basic orthogonal translations reverse the orientation of the surface. Extending and trimming this presentation appropriately, the dual octahedral structure is revealed (Figure 2, center).

Conway et al have designated the prime space group as 2+:2. This group is also known for its display of a tetrahedral structure similar to those of a diamond crystal. However, in the catalogue of plenary groups, Conway et al also reference the mucuboctahedron under 4+:2 described as the doubled tetrad group ([2], p. 309). This seems to relate to whether the translations preserve or do not preserve orientation.

These triply periodic polyhedra are a rich resource for the study of symmetries. By way of example, here are some things to notice about this model:

1. The model has a covering by a symmetric tiling of the hyperbolic plane by hexagons and squares.
2. The model exhibits full tetrahedral symmetries (ignoring the colors and designs) and a subgroup of the symmetries of the cube and the octahedron.
3. The model has the symmetries of a face-centered cubic lattice or a cubic lattice, depending upon whether or not you allow reversals in the surface orientation.
4. We can see examples of all four classes of orientation preserving and reversing symmetries of a surface embedded in Euclidean 3-space.
   (a) A pure translation along one edge in the cubic paper model is a symmetry which reverses the orientation of the surface, and of normals to the surface.
   (b) A reflection about a plane containing one of the hexagonal openings between chambers combined with a 60 degree rotation, reverses space orientation and orientation within the surface, without reversing normals to the surface.

(c) A reflection about a plane containing one of the hexagons or squares combined with an appropriate rotation reverses the space orientation and the normals, without reversing the orientation within the surface.

(d) Many other rotation and translation symmetries (and the identity, of course) will preserve all three types of orientation.

Acknowledgements

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