Iterating Borromean Rings on a Sphere

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Abstract

Borromean Rings are a simple yet interesting set of three rings intertwined so that if any one ring is cut and removed, then the other two rings separate into unlinked rings. Here we search for interesting ways to iterate Borromean Rings to obtain artistically interesting weavings upon a sphere which continue to retain this property. We then explore the symmetries and antisymmetries of the resulting constructions.

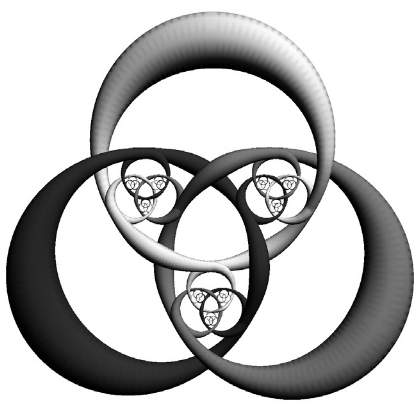

Figure 1.
Iterated Borromean Rings

In [1], the author iterated Borromean Rings to create the Rings shown in Figure 1. In this paper we start with the Borromean Rings rearranged into a spherical representation, Figure 4, thus transforming the image from planar to spherical. This concept was suggested by a reviewer of the previous paper. These constructions were motivated by Robert Fathauer's paper *Fractal Knots Created by Iterative Substitution* and his artwork *Infinity* [3,4].

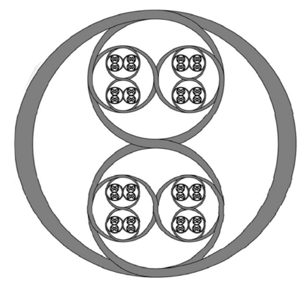

Figure 2.
Fathauer's *Infinity*

Borromean Rings

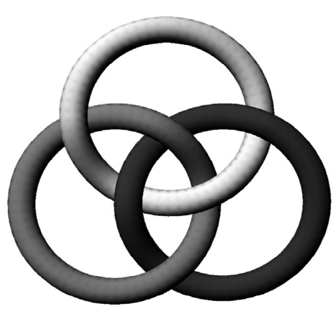

Figure 3.
Borromean Rings

Borromean Rings consist of three rings linked together and yet when any single ring is removed the other two rings become unlinked. Figure 3 shows the most common representation of the Borromean Rings. This name comes from their use in the Borromeos' family crest in the fifteenth century. Although Peter Tait, in 1876, was the first mathematician to study these rings, the name Borromean Rings was not used until 1962 in a paper by Ralph Fox. Please see [5] for an excellent discussion of the history of Borromean Rings. Unsurprisingly, the discussion includes examples which precede their use by the Borromeo family.

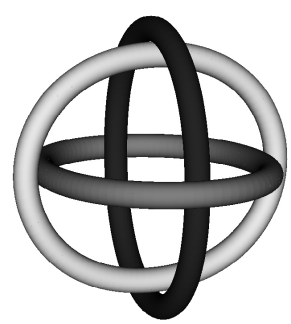

Figure 4.
Spherical Representation of the Borromean Rings

Three Spherical Constructions

Our goal is to iteratively splice copies of the planar representation of the Borromean Rings, shown in Figure 3, into each of the eight faces of the spherical representation, shown in Figure 4. Notice that the planar rings are colored correctly to splice into the top left face of Figure 4 so that the colors match the edges. However, we must transform these rings to correct the coloring before splicing into the top right face of Figure 4. We can transform the rings either by flipping the rings over as shown in Figure 5a or by reflecting as shown in Figure 5b. Although either could be used, we shall choose to use the mirrored. Thus, all planar rings are either copies of Figure 3, which we shall call *positive*, or Figure 5b, which we shall call *negative.* This choice has the desirable property that crossings are consistently of the form:

light* > *medium* > *dark* > *light

where ">" means ***crosses above***. Notice that relative to the surface of this paper, the rings in Figure 4 do not cross over each other in this fashion. If, however, we view each crossing as if we are looking down onto the surface of an enclosed sphere toward its center, then every crossing is one of the three listed in the above inequality. Thus, for every construction below, the light loop is over the medium is over the dark is over the light when viewed relative to the surface of the sphere.

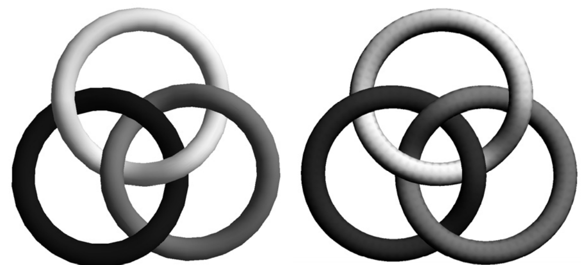

Figure 5.
a. *Planar Flipped* ***b.*** *Planar Mirrored*

We are now ready to cut the rings in Figure 4 at the midpoints of each edge and splice in four positively oriented Borromean Rings and four negatively oriented Borromean Rings to obtain the sphere shown in Figure 6. This now leaves choices on how to iterate this process.

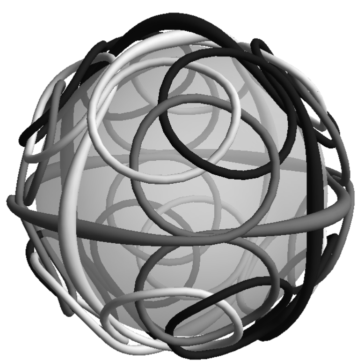

Figure 6.
The First Iteration

We can iterate this process, as shown in Figure 7, by repeatedly placing smaller and smaller copies of the Borromean Rings into the triple intersection portion of each triple of rings. In this situation, we continue to place smaller copies of the positively oriented rings into positive faces and negative into the negative. Naturally, the thickness of the rings must shrink in order to attach the next smaller set of rings. In theory, this process could be repeated indefinitely. As can be seen in the figures shown, this process is visually impractical beyond the second, or possibly third, iteration.

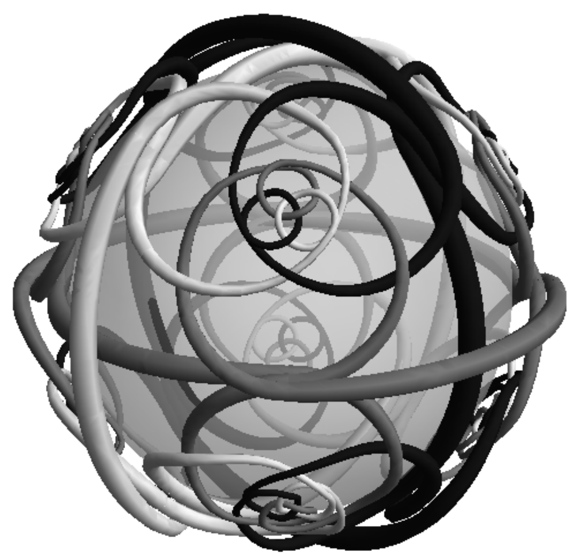

Figure 7.
Central Recursion

Alternatively, as shown in Figure 8, one can place smaller copies of the Borromean Rings into each of the three double intersection portions of each triple of rings. This is the same process that was used to create Figure 1 above. Notice that this process flips the orientation for each successive set of rings. Thus, three negatively oriented Borromean Rings are placed in each positive face and vice-versa.

Backing up, we note that the first iteration, shown in Figure 6, created triple intersections where the planar Borromean Rings were spliced onto the spherical Borromean Rings. We can eliminate these triple intersections by starting with wavy arcs in the spherical representation. This transforms Figure 8 into Figure 9, a weaving without triple points.

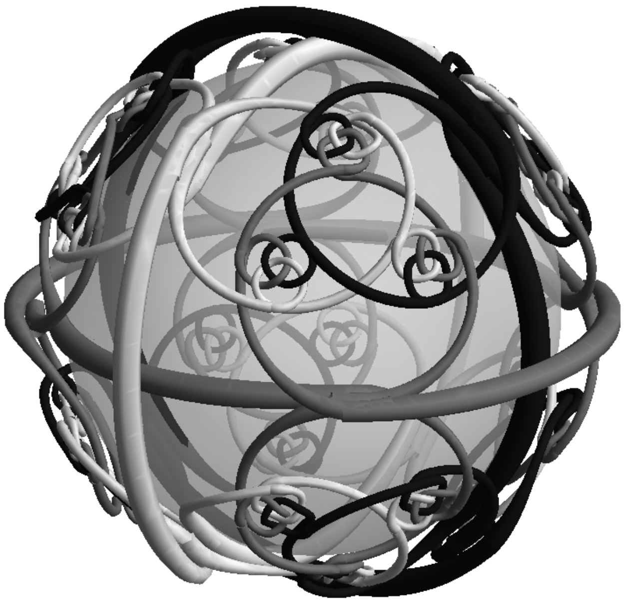

Figure 8.
"Trillium Recursion"

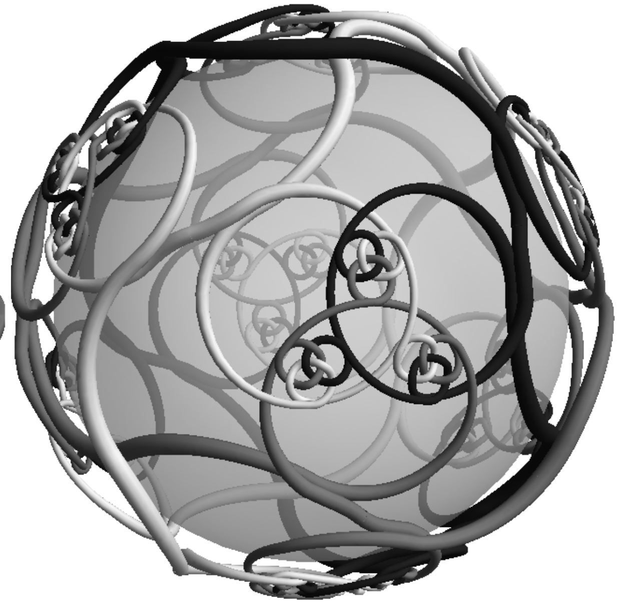

Figure 9.
"Trillium Recursion on Waves"

The Brunnian Property

Borromean Rings are interesting because they are intertwined in such a way so that if any one ring is cut and removed, then the other two rings separate into unlinked rings, each of which is unknotted. This property is called the Brunnian Property in honor of Hermann Brunn. Thus, to verify that each of the above constructions is Brunnian, we must demonstrate two properties: First, that removing any one loop separates the remaining loops into disjoint loops; And, second, that each disjoint loop is unknotted. The first requirement follows from our requirement that the light loop always crosses over the medium which always crosses over the dark which always crosses over the light, keeping in mind that *over* is relative to the surface of the sphere. In Figure 10 we can see that the loops are disjoint when any one loop is cut and removed. And, since each iteration splices small circular rings onto the previous loop, as shown in Figure 11, the iterations do not knot the individual loops.

Symmetries of the Constructions

Let us now consider the symmetries of the three constructions created above. We consider symmetries that preserve color and those that do not. If a transformation is almost symmetrical except that it inverts the crossings relative to the surface of the sphere, then we shall call it *antisymmetrical.*

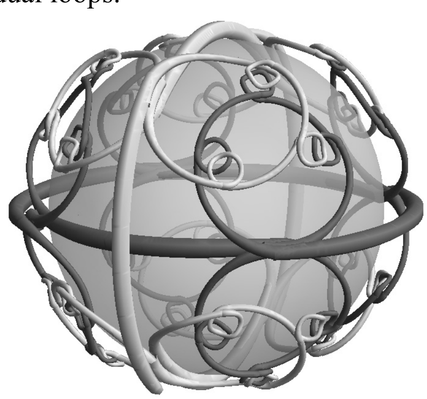

Figure 10.
Pairs of Loops are Disjoint

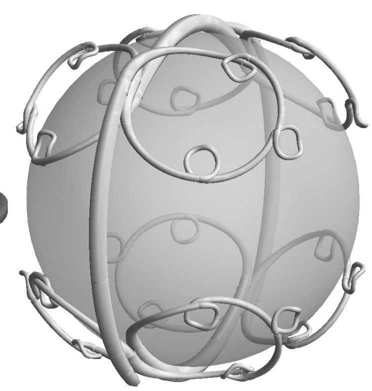

Figure 11.
A Single Loop is Unknotted

We first consider the centers of faces, as shown in Figure 12. We see that we have three-fold rotational symmetry about the center of any face which cycles the three colors.

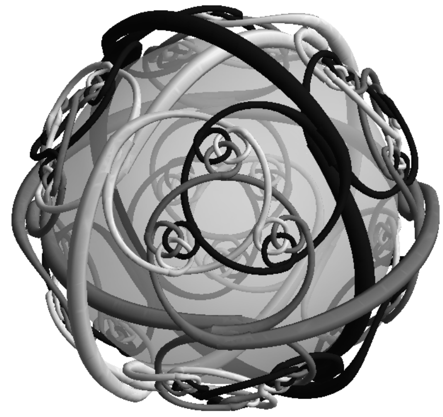

Figure 12.
Three-Fold Rotational Symmetry

We now consider the viewpoint from the midpoint of any side, as shown in Figure 13. We *almost* have bilateral color-preserving symmetry in Figure 13. The left-side, a negative face, is almost a mirror of the right side, a positive face. The only portions that are not mirrored are the triple intersections. Likewise, we *almost* have color switching bilateral antisymmetry between the top hemisphere, above the median line passing through the light triple point, and the bottom hemisphere. Again, the triple intersections are the only portions to fail. Since the middle arc of every triple point is a backwards "S" curve, these constructions cannot have any bilateral symmetries.

Instead, each triple intersection is the center of a color-switching antisymmetric two-fold rotation. This antisymmetry switches two colors and inverts all crossing. This is equivalent to *light > medium > dark > light* switching to the equivalent statement *light < dark < medium < light.* Switching two colors must be antisymmetric since this will switch positive and negative faces and, thus, it must also invert every crossing.

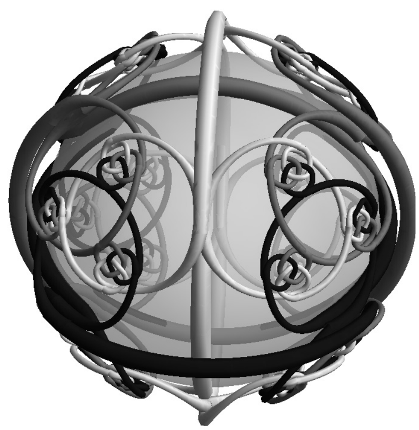

Figure 13.
Antisymmetric Two-Fold Rotation

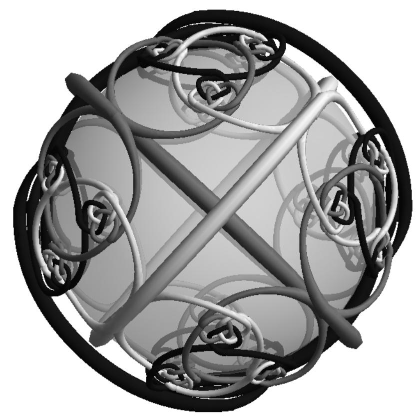

Figure 14.
Two-Fold Rotational Symmetry & Antisymmetric Four-Fold Rotation

Looking down on any vertex of a face, as shown in Figure 14, we have a color-preserving two-fold rotation and we have an antisymmetric color-switching four-fold rotation.

In summary, the three constructions shown in Figure 7, Figure 8, and Figure 9 have the same symmetries: Four axes of three-fold rotational symmetry through the centers of the faces; Three axes of antisymmetric four-fold rotations and two-fold rotational symmetry through the vertices of the faces; And six axes of antisymmetric two-fold rotations through the midpoints of the edges of the faces.

References

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