Following the Footsteps of Daedalus: Labyrinth Studies Meets Visual Mathematics

Kristof Fenyvesi*, Slavik Jablan†, Ljiljana Radović††

*University of Jyväskylä
P.O.Box 35
FI-40014, Finland
fenyvesi.kristof@gmail.com

†The Mathematical Institute
Knez Mihailova 36, P.O. Box 367
11001 Belgrade, Serbia
sjablan@gmail.com

††The Faculty of Mechanical Engineering
Department of Mathematics, A. Medvedeva 14
18000 Niš, Serbia
ljradovic@gmail.com

Abstract

Karl Kerényi, the internationally renowned Hungarian classical philologist and scholar of religion, carried out significant mythological research on the labyrinth, both as a cultural symbol and a specific geometrical structure. In this paper, we summarize Kerényi's main findings and connect his mythological studies with visual-mathematical and ethno-mathematical research to provide a geometrical and topological analysis of these symbolic structures. In our visual mathematical analysis, we propose a very simple "cut and paste" construction of labyrinths based on a system of parallel black and white strips. Then we generalize the concept of the labyrinth and extend it to arbitrary graphs. Finally, we introduce an interesting example for the use of labyrinthine structures in contemporary art and in interactive exhibitions promoting experience-centered education of mathematics.

Introduction: Labyrinth Studies and Archaeology

According to the Greek myths, the skillful craftsman Daedalus created the Labyrinth. The purpose of this special architectural structure was to imprison the Minotaur, the son of Pasiphaë, the wife of the Cretan King Minos. The myth of the Cretan Labyrinth has been a subject of speculations and archaeological, historical, and anthropological research for a long time – just as the visual representations of labyrinthine structures concern not only art historians, but mathematicians too.

Karl Kerényi (1897-1973), the internationally renowned scholar of religion – colleague of Carl Gustav Jung, and a friend and advisor of Thomas Mann – returned time after time to the mythological research of the labyrinths and interpreted them both as cultural symbols and specific geometrical structures. Right from the beginning of his labyrinth studies, Kerényi introduced the labyrinth from three closely interrelated main aspects: 1) he discussed the labyrinth as a mythical construction; 2) as a spiral path that was followed by dancers of a specific ritual; and 3) as a structure that was represented by a spiral line. In his 1941 essay series [1], he summarized the most important concepts of the previous studies and made several original observations and comparisons, which are still widely quoted and referred to in Labyrinth Studies.

With the comparative mythological and morphological analysis of the Babylonian, Indonesian, Polynesian, Australian, Normand, Roman, Scandinavian, Finnish, English, German, and medieval and Greek labyrinth tradition, he has proven the global presence of labyrinthine structures and revealed the artistic and architectural impulse behind the creation of them to rituals and cultic dances when participants followed spiral lines and made meandering gestures and dance-movements. In 1963, Kerényi devoted a lengthy essay to the Greek folk dance [2] and pointed out how the movements of the ancient labyrinth dances were transformed into the main components of the *syrtos*, a dance that is still performed in Greece today. And in his last book written in 1969 [3], where he explored the Cretan roots of the cult of Dionysos, he discussed in depth the labyrinthine and meander-like patterns of Knossos in dance, art, and architecture. ([3], 89-128)

”When a dancer follows a spiral, whose angular equivalent is precisely the meander, he returns to his starting point,” wrote Kerńyi, quoting Socrates from Plato’s dialogue *The Euthydemus*. Socrates speaks (at 291b) there of the labyrinth and describes it as a figure whose most easily recognizable feature is an endlessly repeated meander or spiral line: ”Then it seemed like falling into a labyrinth; we thought we were at the finish, but our way bent round and we found ourselves as it were back at the beginning, and just as far from that which we were seeking at first” ([3], 92-93). According to Socrates’ account, ”the labyrinth is a confusing path, hard to follow without a thread, but, provided one is not devoured at the mid-point, it leads surely, despite twists and turns, back to the beginning.” To explain the development and variations of the structure, Kerényi calls attention to a fresco on the ground floor of the palace of Knossos ([3], 94), which has a complex meander pattern that runs not only in one direction but which also lacks an entrance or exit to the system as a whole. Based on this fresco, Kerényi describes the labyrinth construction ”as an ingenious composition of endless spirals or meanders (depending on whether the drawing is round or angular) on a delimited surface. There resulted a classical picture of this procession, which originally led by way of concentric circles and surprising turns to the decisive turn in the center, where one was obliged to rotate on one’s own axis in order to continue the circuit” ([3], 96). The labyrinths’ ”surprising turns” and the ”decisive turn in their center” is responsible for their symbolic meaning as well. Kerényi sees the labyrinths as depictions of Hades, the underworld, and interprets the structures as narrative symbols which express the existential connection between life and death, between the oblivion of the dead and the return of the eternal living.

From a morphological perspective, Kerényi presupposes the transformation of the spiral to the meander pattern because the straight lines were easier to draw, and so the rounded form was early changed into the angular form. For Kerényi, the meander is the figure of a labyrinth in linear form. In the third to second centuries B.C., as he explains, we find the figure and the word unmistakably related: in the Middle Ages, the labyrinths were also called meanders ([1], 249). Regarding the morphological evolution of the labyrinthine forms in art, architecture, and ornamentation, and taking into account the structure’s complex and generic relationship with meander-like ornaments, we find a detailed description of the connection between meanders and labyrinths in the VIII-IX chapters of W.H. Matthews’ book, *Mazes and Labyrinths* [4].

Although both Matthews and Kerényi made the connection between labyrinths and meanders clear, and there were also comprehensive studies and sophisticated explanations even in the field of archaeology on the technical questions of the geometrical construction problems of the ancient Greek labyrinths, [5] the ornamental evolution of the angular labyrinths were not discussed by any of them in a way that could explain the geometrical development process underlying them. In the following, an example for an alternative, visual mathematical approach is given.

Labyrinth Studies and Visual Mathematics

We have found that the oldest examples of geometrical ornamentation in Paleolithic art were from Mezin (Ukraine) dated to 23 000 B.C.

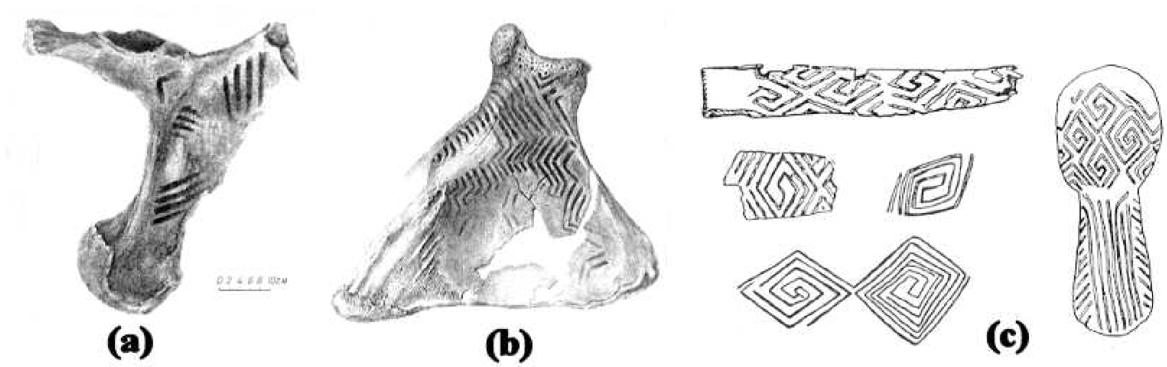

Figure 1 : *Ornaments from Mezin.*

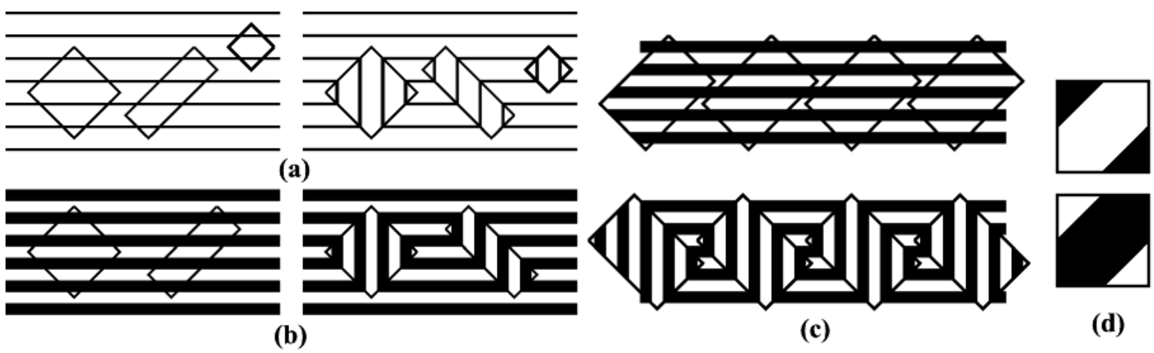

Figure 2 : *(a,b) "Cut and paste" construction; (c) Kufic tiles.*

At first glance, the ornament on the right side of Figure 1a appears to not be significant–it is an ordinary set of parallel lines. The next series of ornaments from Mezin is more advanced. The previously mentioned sets of parallel lines are arranged in friezes and meander patterns that can still be considered as sketches (Figure 1c,1d). However, among them appears the first known meander frieze in the history of mankind, well known under the name "Greek key". Take a set of parallel lines, cut a square or rectangular piece with the set of diagonal parallel lines incident to the first ones, rotate it by 90°, and if necessary translate it in order to fit with the initial set (Figure 2a). More aesthetically pleasing result will be obtained by using the initial set of black and white strips of an equal thickness (Figure 2b).

From meanders to labyrinths

The word "labyrinth" is derived from the Latin word *labris*, meaning a two-sided axe, the motif related to the Minos palace in Knossos. The walls of the palace were decorated by these ornaments while the interior of the palace featured actual bronze double axes. This is the origin of the name "labyrinth" and the famous legend about Theseus, Ariadne, and the Minotaur. The Cretan labyrinth is shown on the silver coin from Knossos (400 B.C.) (Figure 3a).

The simplest natural maze or labyrinth is a spiral meander: a piecewise-linear equidistant spiral. It is defined by a simple algorithm: start from a central point and after every step turn by 90° and continue with the next step, where the sequence of step distances is $1, 1, 2, 2, 3, 3, 4, 4 \ldots$. Tracing this sequence, we have a *maze path*: a simple curve connecting the beginning point (the entrance) with the end point (Minotaur room).

First we consider Simple Alternating Transit mazes, or SAT mazes [6]. A SAT maze is laid out on a certain number of concentric or parallel *levels*. A maze is *simple* if the path makes essentially a complete

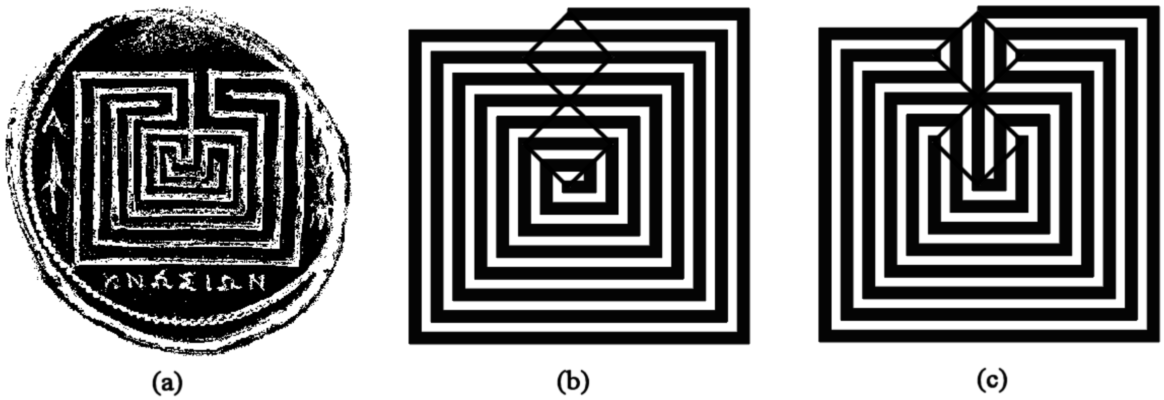

Figure 3 :

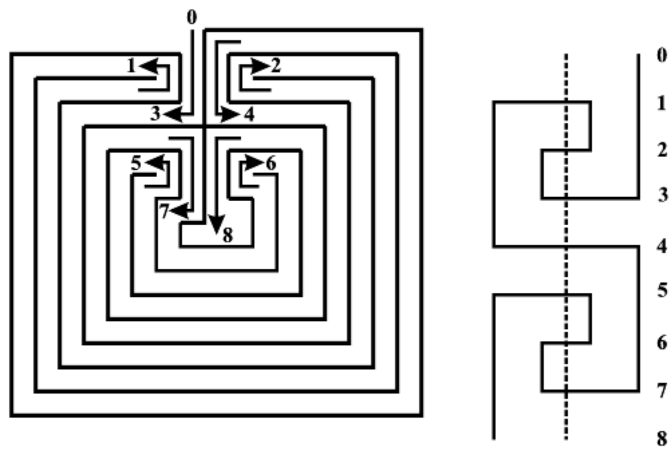

Figure 4 : *Meander permutation and "unrolling" procedure of a maze.*

loop at each level; in particular, it travels on each level exactly once. It is *alternating* if the maze-path changes direction whenever it changes level, and *transit* if the path runs without bifurcation from the outside of the maze to the center.

Definition 1. *An open meander is a configuration consisting of an oriented simple curve, and a line in the plane, the axis of the meander, that cross a finite number of times and intersect only transversally. Two open meanders are equivalent if there is a homeomorphism of the plane that maps one meander to the other [7].*

Most simple alternating transit mazes occur in a spiral meander form with the path leading from the outside to the center. Each such maze can be sliced down its axis and unrolled into an open meander form. Now the path enters at the top of the maze and exits at the bottom: the top level (center) of the maze becomes the space below the open meander form. This process is illustrated in Figure 4 for the Cretan maze. The topology of a SAT maze is entirely determined by its level sequence, i.e., its open meander permutation. The modern study of meanders enumeration problem is inspired by Arnol'd [8]. However, at the same time Warren Smith (1988 Princeton thesis) and Kazakov-Kostov were discovering the same enumeration problem, so it is a nice example of mathematical simultaneity [6]. Meanders occur in polymer physics, map folding, in the study of planar algebras (in particular, Tempereley-Lieb algebra), and in knot theory, where we recently introduced a new concept: meander knots and links [9]. If the intersections along the axis are enumerated by $1, 2, 3, ..., n$ every open meander can be described by a meander permutation of order n: the sequence of n numbers describing the path of the meander curve. For example, the open meander (Figure 4) defining

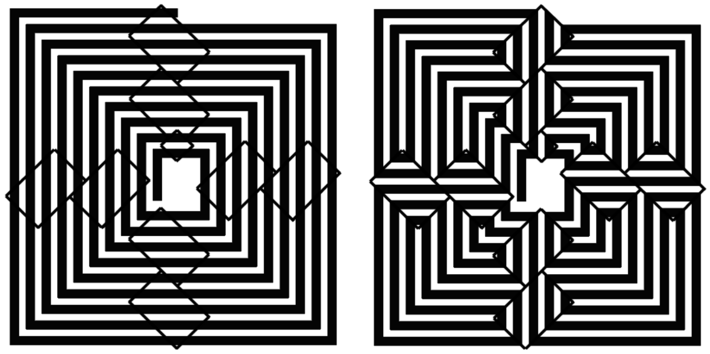

Figure 5 : *"Cut and paste" construction of a complex maze.*

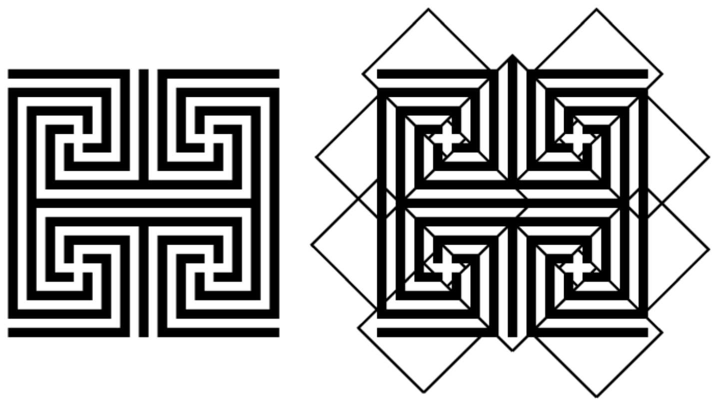

Figure 6 : *Knossos' "dancing pattern".*

the Cretan maze is coded by the meander permutation $(3, 2, 1, 4, 7, 6, 5)$. Hence, the enumeration of open meanders and their corresponding SAT mazes is based on the derivation of meander permutations. For the derivation of open meanders one can use the *Mathematica* program "Open meanders" by David Bevan [10], which we modified in order to compute open meander permutations.

How does one construct a unicursal maze without knowledge of computer programs and topological transformations? Figure 3 shows an elegant way to construct a Cretan maze: draw a black spiral meander (Figure 3b), cut several rectangles or squares, rotate each of them around its center by the $90°$ angle, and place it back to obtain a labyrinth (Figure 3c). Even very complex mazes can be constructed in this way (Figure 5). It is interesting to notice that even Knossos' "dancing pattern", using the shape of a double axe, can be reconstructed in a similar way (Figure 6). So, a non-significant pattern (Figure 1a), an *Op-tile* [11], can be considered as a logo of a Paleolithic designer, from which Mezin ornaments can be created.

Kufic tiles

If in a black and in a white square we construct one diagonal region of opposite color, we obtain two modular elements called Kufic tiles (Figure 2c. From these modular elements, named by S. Jablan *Kufic tiles*, you can create square Kufic letters and Kufic scripts: the writing of letters, names, and texts, where all black and white lines are of the same width. Kufic tiles can be used as pixels for the construction of all possible mazes.

Professor Donald Knuth, a master of computer art and the author of the program TeX, designed new TeX-fonts from our Kufic tiles (`http://www-cs-faculty.stanford.edu/~knuth/graphics.html`). The text and the Knossos labyrinth in Figure 7 are typed using his fonts.

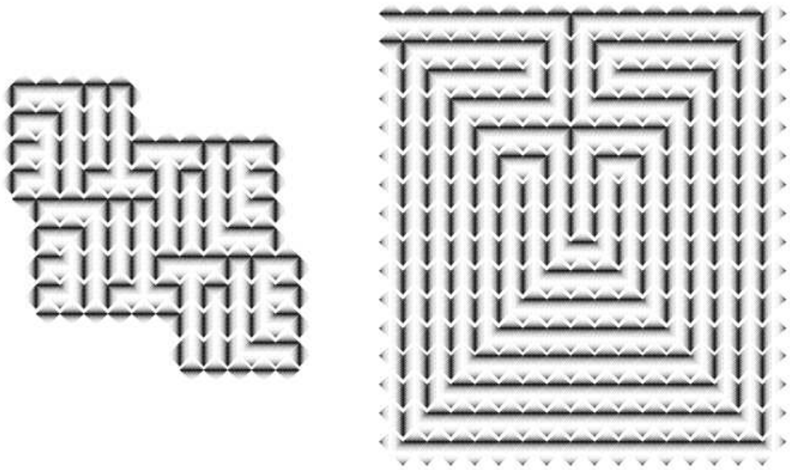

Figure 7 : *Text written with Knuth's fonts based on our Kufic tiles.*

Labyrinths revisited

Every labyrinth has the same goal: to uniformly cover some area and provide a path for Theseus to get to the Minotaur's room. Traditionally, most labyrinths are symmetrical, in a form of a system of concentric squares or its octagonal or circular topological variations, where, as usual, the Minotaur is in the center [12]. In some cases, the creators of labyrinths tried to change these rules: at least to break symmetry or change the standard shape of a labyrinth (Figure 9). However, these attempts are sometimes followed by a small imperfection (see the gray isolated squares in Figure 9). So, during the last few thousand years, the labyrinth game became boring, probably not only for Theseus and the Minotaur. One possibility to make it more interesting is to translate just the first sentence of this paragraph, a definition of a labyrinth, to the language of mathematics: in a given plane graph G (e.g., grid graph $G(m,n)$) a labyrinth is a Hamiltonian path in G connecting the entrance: a vertex belonging to an external face with some internal vertex. This gives a chance to the Minotaur to hide in some part of the graph (Figure 8 a), or even to survive: to find a safe place, vertex in G that is not reachable from the entrance by any Hamiltonian path. Sitting in this safe place, he can think about the mathematical problem: how many safe vertices are in G for a fixed entrance point. After exhausting all possibilities, maybe he can propose to Theseus to play a new game in some 3D-labyrinth (Figure 8, where is shown only the path in this 3D-labyrinth, and internal and external planar faces, the walls of the labyrinth, are omitted), or even in a n-dimensional grid graph.

Some sophisticated labyrinthine solutions are shown in Figure 9 and Figure 11. These works demonstrate that the expressive power and cultural significance of labyrinths did not decrease throughout the history. This high cultural and mathematical potential of labyrinthine structures also can be employed in education of mathematics and arts. At the previous *Bridges* Conferences there were many remarkable examples of the simple and ingenious educational implementation of labyrinths by Samuel Verbiese [14]. The authors of this article also successfully experiment with the use of labyrinths in the playful education of visual mathematics. As it was shown in works by Ben Nicholson and S. Jablan (Figure 10), labyrinths can be composed from modular tiles: *Versi-tiles* (the name proposed by B. Nicholson), or *Op-tiles* [11]. Our educational tools based on this idea proved to be very popular elements of our interactive math-art exhibits and workshops, organized by the Experience Workshop Math-Art Movement, and let us demonstrate a lot of surprising features of complex structures based on modularity (Figure 10).

In contemporary art, in the work of Peter Kogler [13], we can see a skillful placing of a maze on a non-flat surface–the multiple walls of a room, showing that labyrinths are topological structures (Figure 11), and covering of a floor with the anamorphic view of a labyrinth which highlights very clearly the "rotated

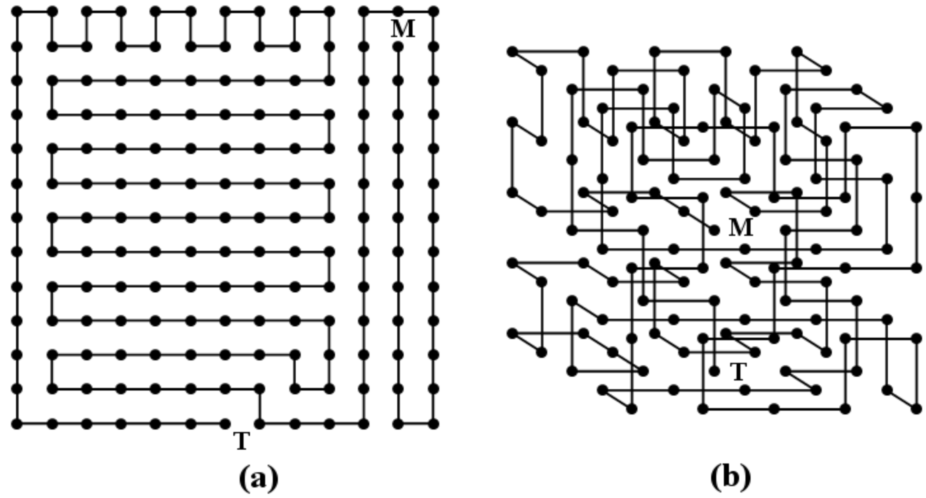

Figure 8 : *(a) Labyrinth path in* $G(13,13)$*; (b) 3D-labyrinth path in a grid graph* $G(5,5,5)$*, where T stands for Thaeseus, and M for Minothaur.*

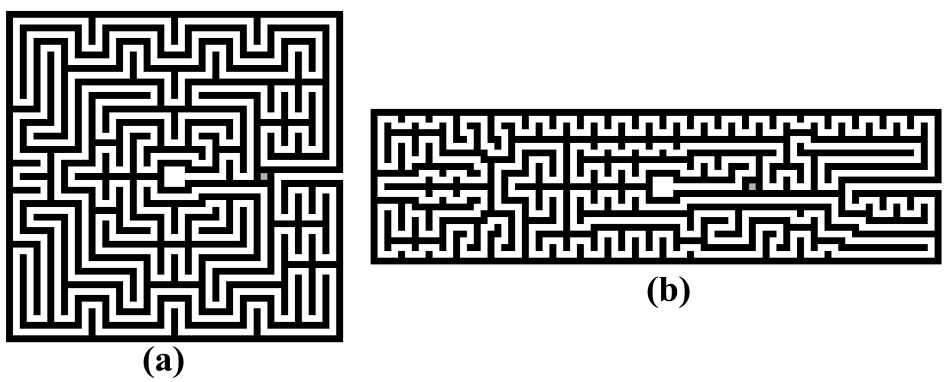

Figure 9 : *(a) Labyrinth from Basilique Notre-Dame, St. Omer; (b) labyrinth from Cathédrale St.-Franois-de-Sales, Chambéry.*

Figure 10 : *Modular labyrinth from Bridges conference in Pécs 2010 (photos by Norbert Horváth.)*

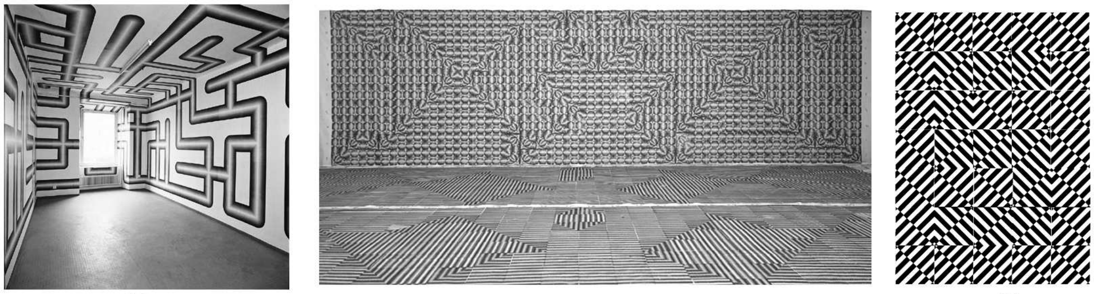

Figure 11 : *(a),(b) Works by P. Kogler; (c) "Letter S" by S. Jablan.*

rectangles" described earlier (Figure 11b). Intentionally used, this anamorphic property of labyrinths offers some new possibilities explored in "horizontal graphics" by S. Jablan (Figure 11c).

References

[1] K. Kerényi, *Labyrinth-Studien. Labyrinthos als Linienreflex einer mythologischen Idee*, In: Humanistische Seelenforschung. Langen Müller - München, Wien, 1966, 226–273.

[2] K. Kerényi, *Vom Labyrinthos zum Syrtos. Gedanken über den griechischen Tanz*, In: Humanistische Seelenforschung. Langen Müller, München, Wien, 1966, 274–288.

[3] K. Kerényi, *Dionysos: Archetypal Image of Indestructible Life*, Ralph Manheim trans., Princeton University Press, 1976.

[4] W.H. Matthews, *Mazes and Labyrinths*, Longmans, Green and Co., London, 1922.

[5] J.L. Heller, *A Labyrinth from Pylos?*, American Journal of Archaeology, Vol. 65, No. 1 (Jan., 1961), pp. 57–62.

[6] T. Phillips, *The Topology of Roman Mosaic Mazes*, Leonardo, Vol. 25, No. 3/4, 1992, pp. 321–329, (see also `http://www.math.sunysb.edu/~tony/mazes/index.html`, `http://www.math.sunysb.edu/~tony/mazes/otherapps.html`), Accessed 19.03.2013.

[7] M. La Croix, *Approaches to the Enumerative Theory of Meanders*, `http://www.math.uwaterloo.ca/~malacroi/Latex/Meanders.pdf`, 2003, Accessed 19.03.2013.

[8] V.I. Arnol'd, *The branched covering $CP^2 \to S^4$, hyperbolicity and projective topology*, Sibirsk. Mat. Zh. **29** (1988), 5, 36–47, 237.

[9] S. Jablan, Lj. Radović, *Meander knots and links*, arXiv:1302.1472v1 [math.GT], `http://arxiv.org/abs/1302.1472`, Accessed 19.03.2013.

[10] D. Bevan, *Open Meanders*, `http://demonstrations.wolfram.com/OpenMeanders/`, Accessed 19.03.2013.

[11] S.V. Jablan, *Symmetry, Ornament and Modularity*, World Scientific, Singapore, 2002.

[12] P. Mudge, *Unicursal Mazes*, `http://www.mudge.screaming.net/MazeIndex.htm`, Accessed 19.03.2013.

[13] Peter Kogler, `http://www.kogler.net/index.php`, Accessed 19.03.2013.

[14] Samuel Verbiese, `http://gallery.bridgesmathart.org/exhibitions/2011-bridges-conference/samuelverbiese`, Accessed 19.03.2013.