Iterative Arrangements of Polyhedra – Relationships to Classical Fractals and Haüy Constructions

Robert W. Fathauer
Tessellations Company
3913 E. Bronco Trail
Phoenix, AZ 85044, USA
E-mail: rob@tessellations.com

Abstract

This paper exhibits and explains esthetically-pleasing constructions using scaled-down polyhedra that have been iteratively arranged on the faces of a starting polyhedron. Sierpinski triangles usually arise when half-scale polyhedra are iteratively arranged on three faces meeting at a vertex. In contrast, a regular array results when half-scale polyhedra are iteratively arranged on four faces meeting at a vertex. The convex hulls of such constructs are the duals of the starting polyhedra for a variety of polyhedra. These arrangements can be thought of as generalized Haüy constructions using a scaling factor less than one. One half is shown to be a special number for such scalings. When arrangements are made about vertices with five faces, a scaling factor of the square of the Golden mean results in a fractal that can be described as a Sierpinski pentagon.

1. Haüy Constructions and a Fractal Crystal

Three-dimensional fractals can be created by iteratively arranging successively smaller generations of polyhedra around a starting polyhedron. In what is probably the first example of computing what one of these structures might look like, William Gosper and Hans Morovec in the 1970’s found out that an iterative arrangement of tetrahedra in a 3-dimensional analog to the Koch Snowflake forms a cube [1]. Some of the structures described here have been reported previously, and other closely-related structures have been described. A fractal arrangements of cubes is described in Reference 2, arrangements of octahedra in References 3 and 4, dodecahedra in Reference 5, and stellated dodecahedra in Reference 6.

We showed previously that arranging half-scale cubes by centering the smaller cubes on the faces of larger cubes leads to a “fractal crystal” with an octahedron as its convex hull and which exhibits myriad Sierpinski triangles [7], as shown in Figure 1. All of the figures in this paper, with the exception of Figures 7-9, were generated in Mathematica.

Over 200 years ago, as part of his investigations into crystallography, René Just Haüy described a technique to obtain several other polyhedra by arranging cubes in layers [8]. The simplest example is the construction of an octahedron, for which six cubes are placed on the faces of a starting cube, then additional cubes are placed on the faces of those cubes, etc., as shown in Figure 2. The parallel to the construction of Figure 1 is clear, and the construction in Figure 1 can be thought of as the $s = 1/2$ case of a generalized Haüy construction with variable scaling factor s.

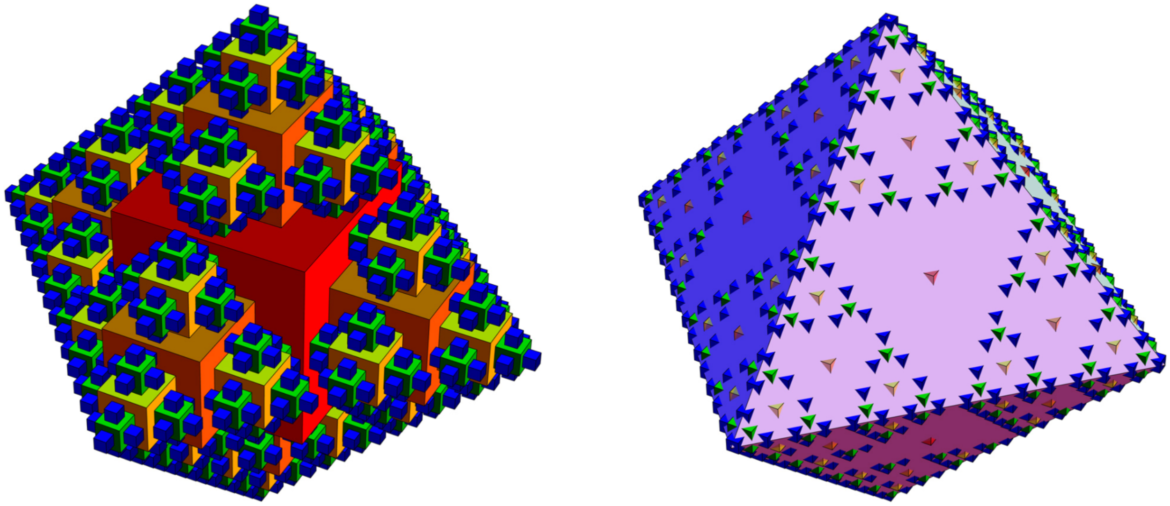

Figure 1: *A iterated arrangement of cubes constructed through five generations with a scaling factor of 1/2. The octahedral shell at right illustrates the Sierpinski triangles formed by the outermost smallest-generation polyhedra.*

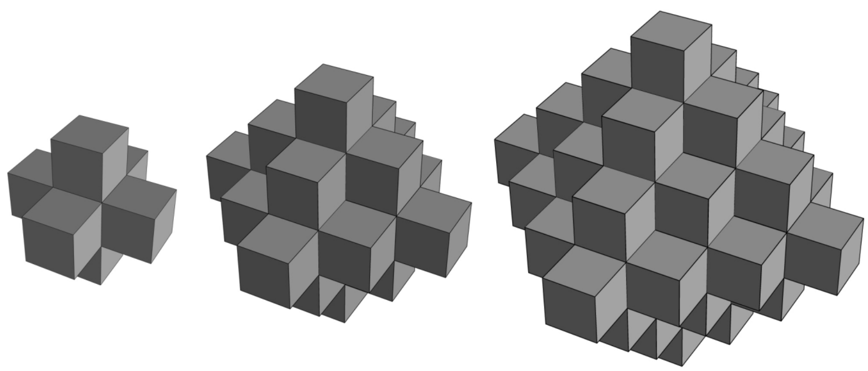

Figure 2: *The second through fourth steps in an Haüy construction of cubes leading, in the limit, to an octahedron.*

Examining other scaling factors reveals that 1/2 is a special number for which cubes that meet always do so in a face-to-face fashion, without partial penetration. Another special scaling factors is 1/3, which results in the outermost edges of cubes lying in planes that define a rhombic dodecahedron, as shown in Figure 3.

3. An Infinite Family of Prism Constructions

Like the cube, the hexagonal prism is a space-filling polyhedron; i.e., multiple copies can be used to tile three-dimensional space. For such polyhedra, Haüy-type constructions analogous of that of Figure 2 can be carried out, in which a polyhedron is placed on each face of the starting polyhedron, etc. For the hexagonal prism case, this will lead in the limit to a hexagonal dipyramid, as shown in Figure 4. If a scaling factor of 1/2 is used, the structure shown in Figure 5 is obtained. This construction has as its convex hull a hexagonal dipyramid, the faces of which exhibit elongated Sierpinski triangles.

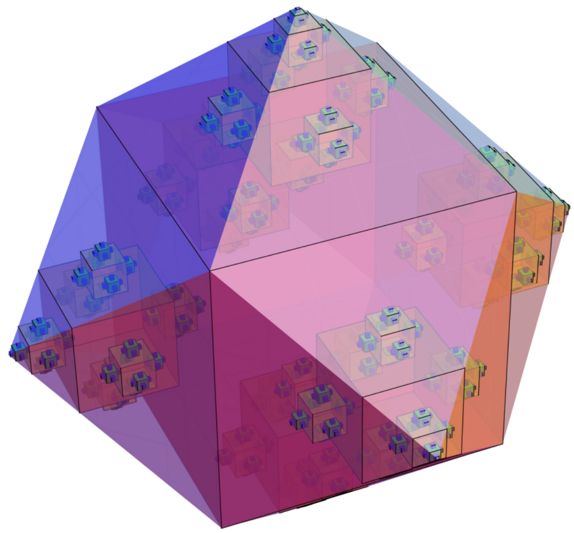

Figure 3: *An arrangement of cubes with a scaling factor of 1/3, along with a partially transparent rhombic dodecahedron in whose faces edges of cubes lie.*

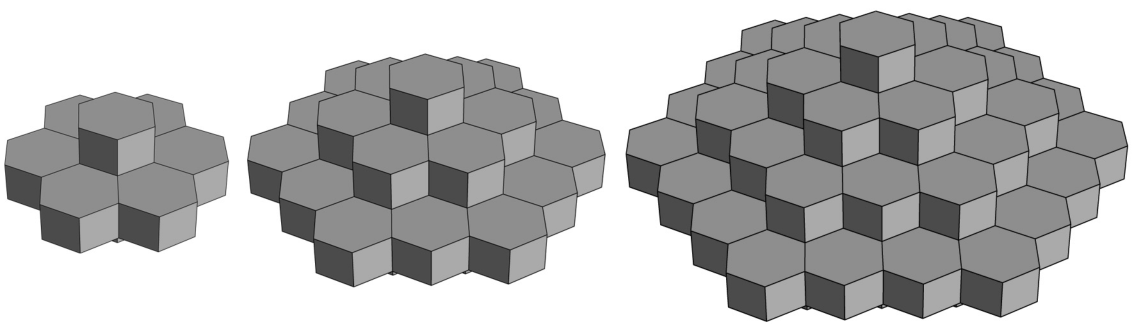

Figure 4: *An Haüy construction of hexagonal prisms that forms, in the limit, a hexagonal dipyramid.*

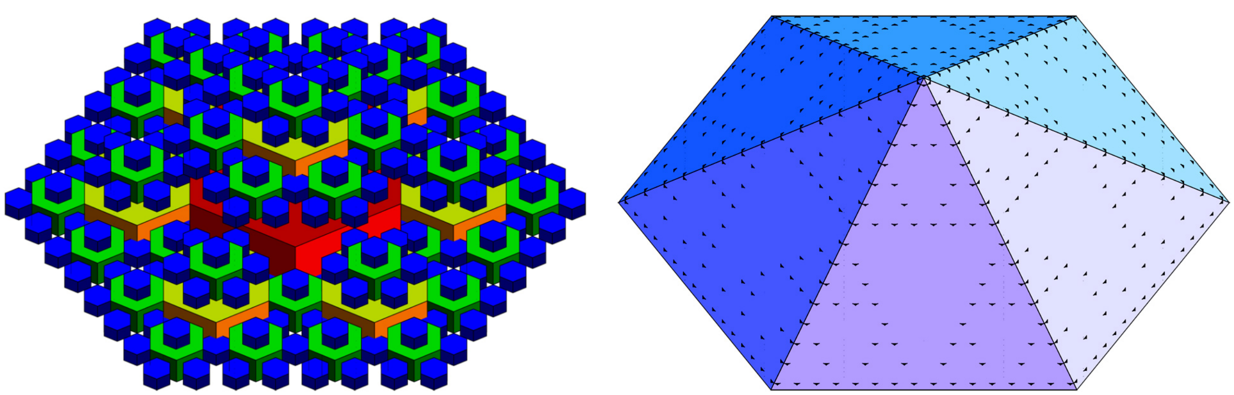

Figure 5: *An iterative construction of half-scale hexagonal prisms through four generations at left, and five generations at right with a hexagonal dipyramid shell illustrating the Sierpinski triangle character of the outer layer.*

The octagonal prism is not space a filling polyhedron, so an Haüy-type construction isn't possible. However, a construction using half-scale prisms is possible. While some partial overlaps occur in the inner region, the outer portion does not have this problem. Focusing on it, a construction is obtained with an octagonal dipyramid as its convex hull, again exhibiting elongated Sierpinski triangles, as shown in Figure 6.

It is apparent that an analogous construction will be obtained for decagonal prisms, dodecagonal prisms, etc. If the construction of Figure 1 is thought of as a collection of square prisms with a square dipyramid as its convex hull (rather than cubes with an octahedral convex hull), it is seen to be the first of an infinite family of analogous constructions. Note that prisms with an odd number of square faces do not follow this pattern as a result of the fact that the prisms will occur in two orientations, preventing such an orderly progression as smaller generations are added.

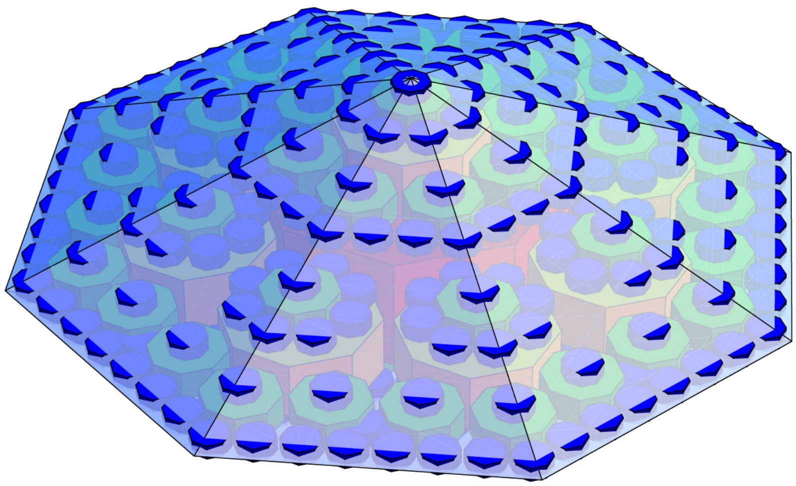

Figure 6: *An iterative construction of half-scale octagonal prisms through four generations, along with a partially-transparent octagonal dipyramid shell.*

4. Fractal Arrangements on a Face of the Convex Hull

In order to understand why the faces of the above constructions exhibit Sierpinski triangles, consider a vertex of a starting polyhedron where three identical faces meet, as shown at left in Figure 7. If half-scale polyhedra of the same type and orientation are placed in the middle of each face, these three polyhedra will form an equilateral triangle defining a plane. Looking in the direction normal to the plane, the next layer of smaller polyhedra will form three equilateral triangles, as shown in the second figure from the left. Two more iterations are shown, from which it can be seen that a Sierpinski triangle is forming.

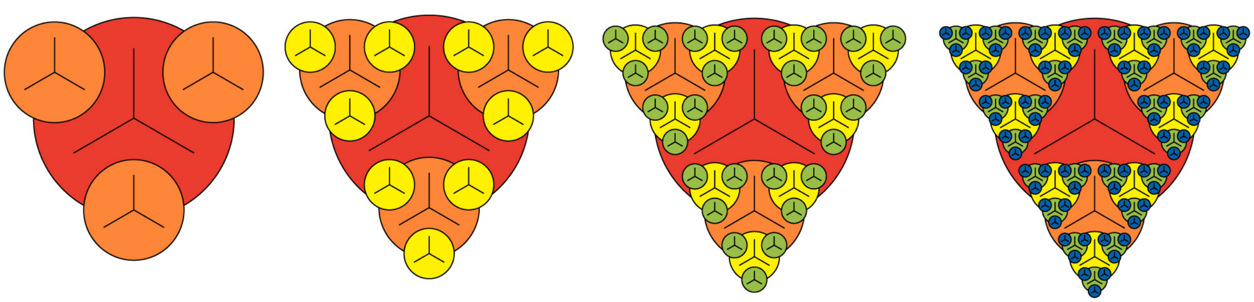

Figure 7: *Arrangement of smaller polyhedra around a vertex where three faces meet.*

If the three faces of the starting polyhedron are not all of the same type, for example in the case of a hexagonal prism, this will have the effect of distorting the starting triangle. However, the plane defined by the three second-generation polyhedra will still be parallel to the plane in which the nine third-generation polyhedra lie, etc. The arrangement will then exhibit distorted Sierpinski triangles, as seen in Figures 5 and 6 above.

If we consider a vertex at which four identical faces meet, the four second-generation polyhedra will form a square. With a scaling factor of 1/2, additional iterations form a regular array of polyhedra, as shown in Figure 8. This explains the result reported previously for iteratively arranging half-scale octahedra about a central octahedon [7], where a structure forms with regular arrays of small octahedra defining the square faces of a cube.

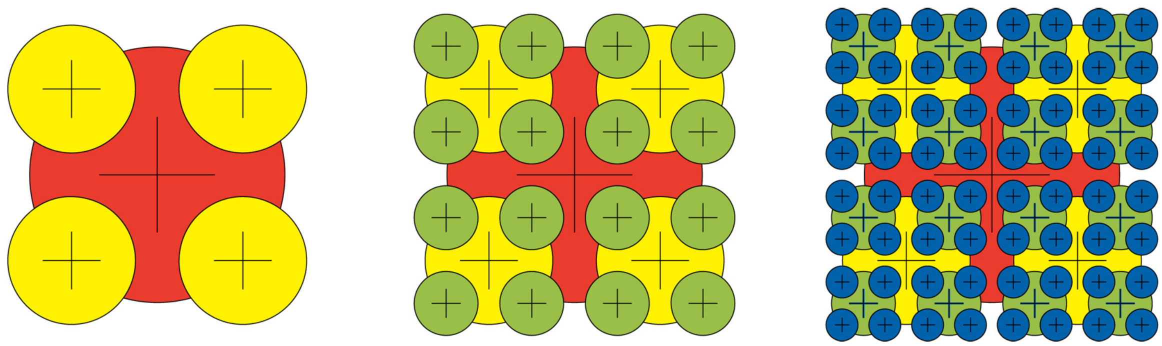

Figure 8: *Arrangement of smaller polyhedra around a vertex where four faces meet.*

If five half-scale polyhedra are arranged about a vertex at which five identical faces meet, scaling by half results in partial overlaps of second-generation polyhedra. Using the square of the Golden Mean as a scaling factor produces a more orderly arrangement, as shown in Figure 9. In this case, a ring of ten uniformly-spaced polyhedra forms. Further iteration results in a structure that has been described as a Sierpinski pentagon [9], with fractal curves developing that have been referred to as pentakoch curves [10].

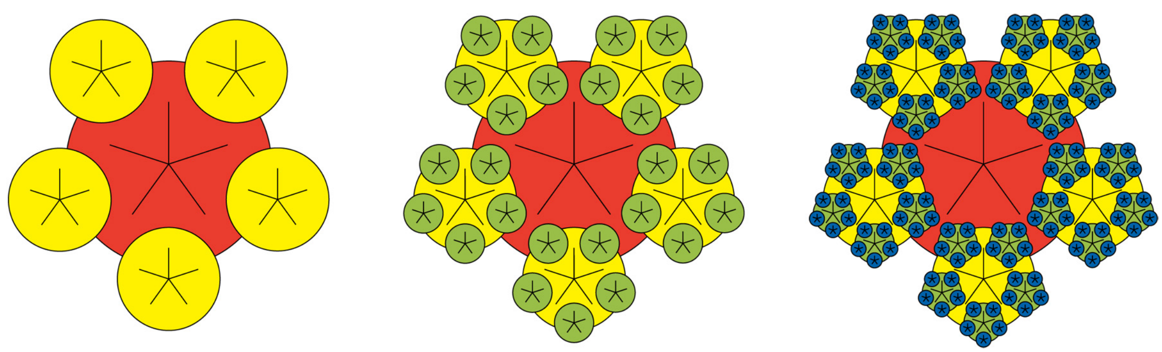

Figure 9: *Arrangement of smaller polyhedra around a vertex where five faces meet.*

5. Further Examples

The rhombic dodecahedron is a space-filling polyhedron with the cuboctahedron as its dual. An Haüy-like construction can be made using this polyhedron, with concentric shells of rhombic dodecahedra placed around a central rhombic dodecahedron. In the limit this construction yields the cuboctahedron. For the half-scale case, this polyhedron serves as an interesting test case because it has vertices where three rhombic faces meet and vertices where four rhombic faces meet. The resulting construction, carried through four generations, is shown in Figure 10. In the directions defined by vectors pointing from the center of the starting polyhedron outward and passing through four-valent vertices, square faces form with regular arrays of polyhedra. In the directions defined by vectors passing through three-valent vertices, equilateral-triangle faces form that exhibit Sierpinski triangles. The convex hull of the construction is a cuboctahedron.

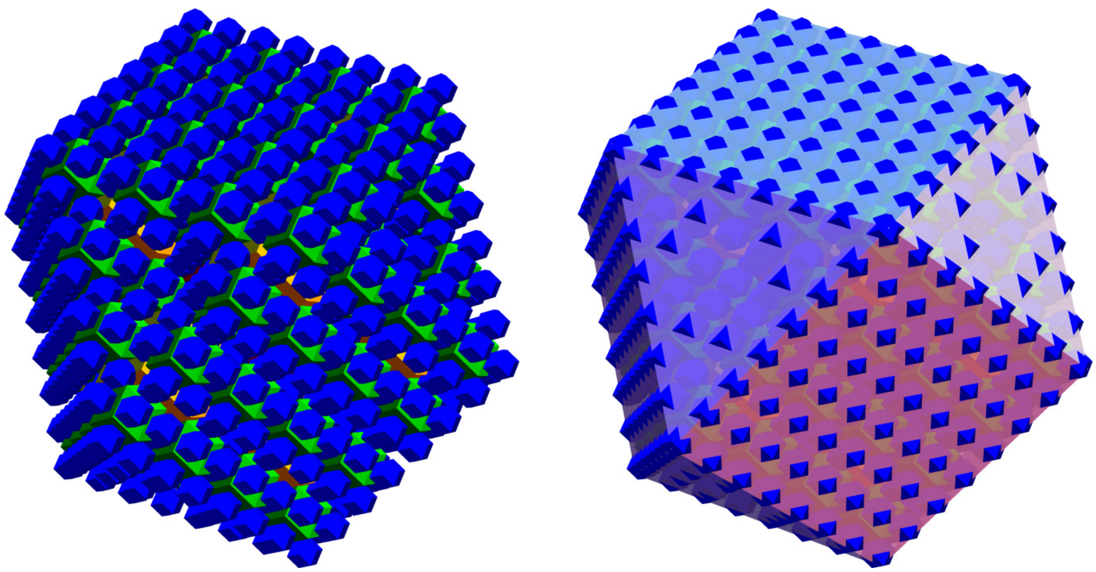

Figure 10: *Iterative arrangement of half-scale rhombic dodecahedra carried through four generations. The cuboctahedron shell at right illustrates how the outermost layer of polyhedra exhibit regular square arrays on the square faces and Sierpinski triangles on the triangular faces.*

The icosahedron is a polyhedron with five identical faces meeting at each vertex, so it serves as good test case for the arrangement illustrated in Figure 9. Its dual is the dodecahedron. Arrangements of both these polyhedra were examined. In the case of the dodecahedron, arranging half-scale dodecahedra around a central dodecahedron results in a construction with an icosahedron as its convex hull and Sierpinski triangles on each of the faces, as shown at left in Figure 11. For the icosahedron, arranging icosahedra scaled by the square of the Golden mean (≈ 0.382) yields a construction with a dodecahedron as its convex hull and the expected Sierpinski pentagon faces, as shown at right in Figure 11.

A final example shows that the situation is not always so straightforward. The truncated octahedron is a space-filling polyhedron that has as its dual the tetrakis hexahedron. An Haüy-type construction created by adding successive layers of truncated octahedra is shown in Figure 12. While one might expect the structure to form a tetrakis hexahedron in the limit, it actually forms a rhombic dodecahedron.

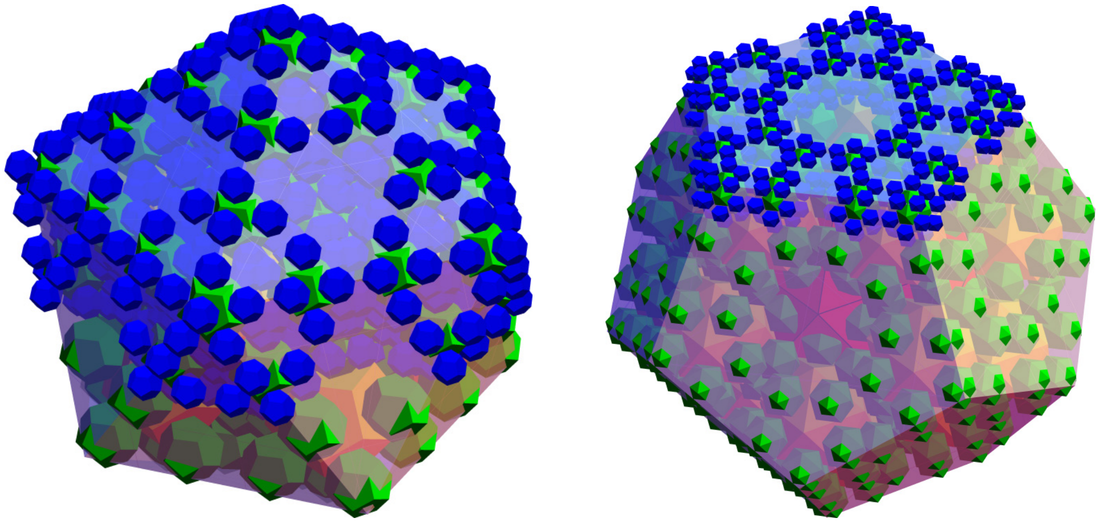

Figure 11: *Iterative arrangement of dodecahedra and icosahedra with partial fourth generations of polyhedra and partially-transparent shells that are the duals of the starting polyhedra.*

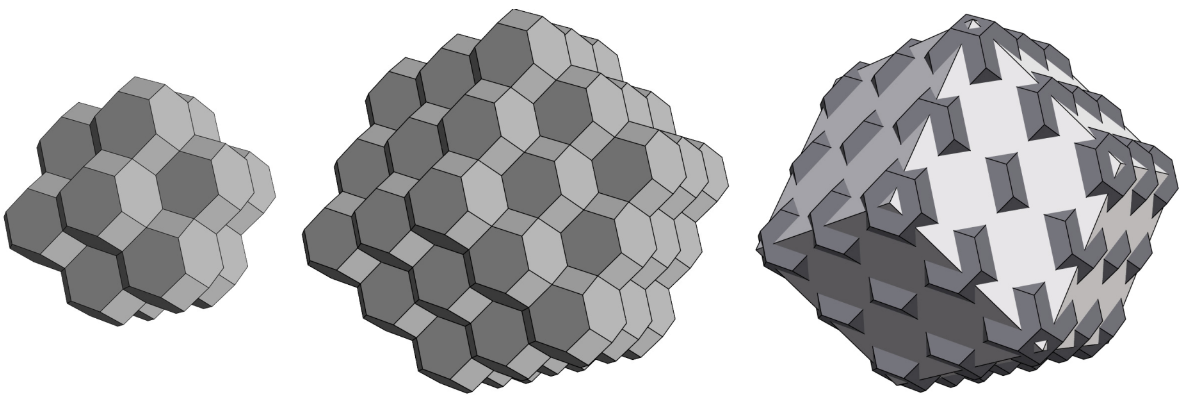

Figure 12: *An Haüy construction of truncated octahedra that forms, in the limit, a rhombic dodecahedron.*

This result can be understood by examining the first step in constructing the half-scale case, shown at left in Figure 13. There are three faces meeting at each vertex, such as the one marked by a black dot in the figure. When the plane is located that contains the three polyhedra placed on those faces, it turns out that a fourth polyhedron also lies in that plane. These four form a rhombus, and a rhombic face forms in each direction from the center of the starting polyhedron out toward the midpoint of each edge shared by two hexagons. The convex hull of the half-scale construction is a rhombic dodecahedron, and the outer layer of polyhedra forms regular rhombic arrays on each face.

6. Conclusions

We have shown structures built from scaled-down polyhedra that have been iteratively arranged on the faces of a starting polyhedron. Several of the examples shown are space-filling polyhedra, and for these the resulting constructions have been compared to Haüy constructions. The convex hulls of both the Haüy-type constructions and the half-scale constructions are in most cases the duals of the starting polyhedra. A Sierpinski triangle arises when half-scale polyhedra are iteratively arranged on three faces meeting at a vertex, as long as no additional polyhedra lie in the plane defined by those three polyhedra. In contrast, a regular array results when half-scale polyhedra are iteratively arranged on four faces meeting at a vertex, as long as those four polyhedra lie in a plane. When arrangements are made about vertices with five faces, a scaling factor of the square of the Golden mean results in a fractal that can be described as a Sierpinski pentagon.

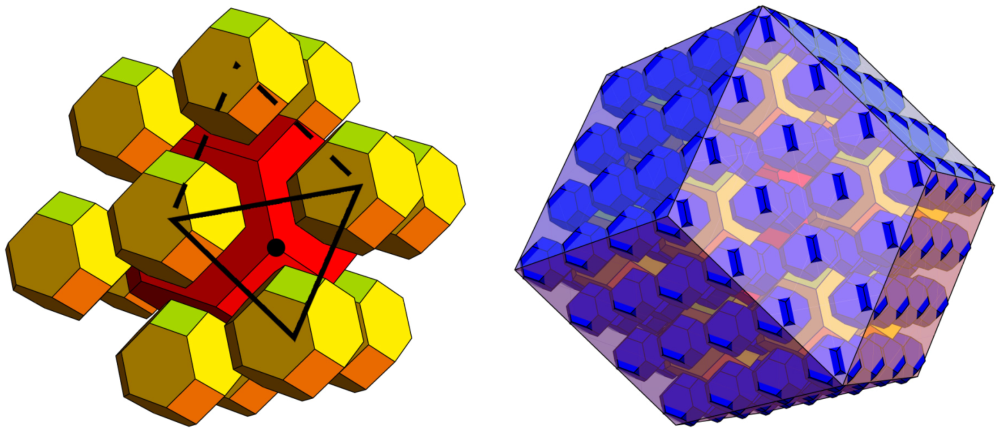

Figure 13: *An iterative construction of truncated octahedra, at left through two generations. The solid and dashed triangles together mark four polyhedra that lie in a plane. The construction is shown at right through three generations, along with a rhombic dodecahedron shell illustrating how each face exhibits a regular rhombic array of polyhedra.*

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