From Möbius Bands to Klein-Knottles

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Abstract

A construction of various immersed Klein bottles that belong to different regular homotopy classes, and which thus cannot be smoothly transformed into one another, is introduced. It is shown how these shapes can be partitioned into two Möbius bands and how the twistedness of these bands defines the homotopy type. Some wild and artistic variants of Klein bottles are presented for their aesthetic appeal and to serve as study objects for analysis.

1. Introduction

This paper extends the study of the regular homotopies of tori presented last year in *Tori Story* [17] to the realm of Klein bottles. Topologically, immersed Klein bottles are non-orientable surfaces of Euler characteristic $\chi=0$, with no boundaries or punctures. Surfaces are in the same regular homotopy class if they can be smoothly transformed into one another without ever experiencing any cuts, or tears, or creases with infinitely sharp curvature; however, a surface is allowed to pass through itself. The paper by Hass and Hughes [9] states (pg.103): Corollary 1.3 (James–Thomas): There are $2^{2-\chi}$ distinct regular homotopy classes of immersions of a surface of Euler characteristic χ into $\mathbf{R}^3$. This tells us there should be **four** distinct Klein bottle types that cannot be transformed smoothly into one another. However, I have found no publication that shows representative pictures of those four types. The single-sidedness of these objects also makes it conceptually more difficult to visualize these shapes. But some good discussions and e-mail exchanges with Dan Asimov, Tom Banchoff, Matthias Goerner, Rob Kusner, and John Sullivan helped clarify the situation. All Klein bottle types can be built from fusing together two Möbius strips, and their twistedness plays an important role in the determination of the type of a given Klein bottle. Thus I include this decomposition in the images below and start this exposition with a discussion of these important building blocks.

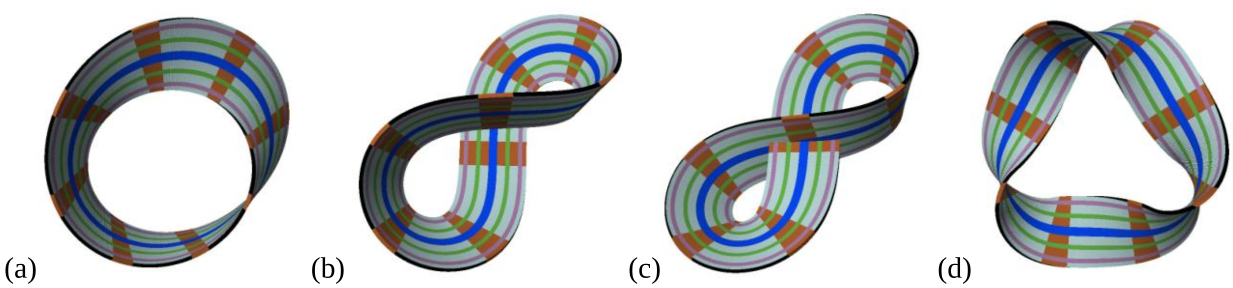
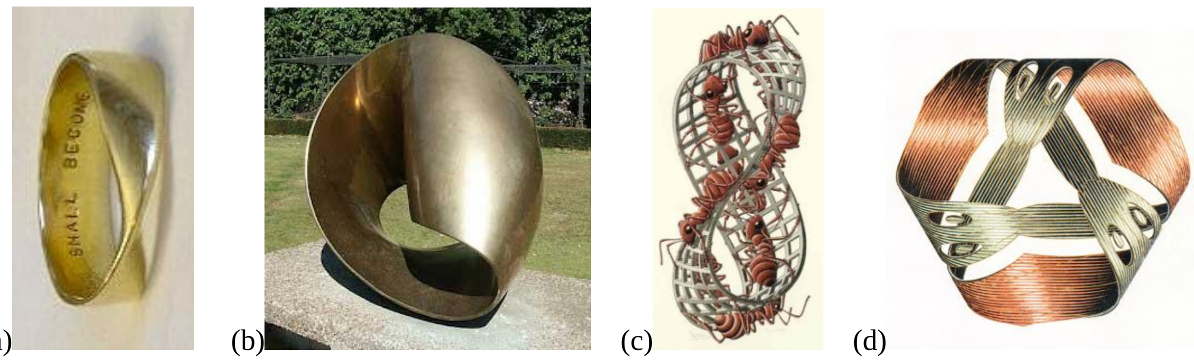


Figure 1: *Deforming a Möbius band of type **ML** and thereby changing its apparent twist compared to a rotation-minimizing sweep from (a) +180° (ccw), to (b) 0°, to (c) –180°, and to (d) –540° (cw).*

2. Möbius Bands

A Möbius band (Fig.1) can be constructed by taking a rectangular domain – for instance a paper strip – and connecting two opposite edges in reversed order, *i.e.*, after executing a 180° flip. For any topological analysis the surface is permitted to pass through itself; thus we can always execute one or more of the *Figure-8 Sweep Cross-over Moves* described in *Tori Story* [17]. This allows us to add twists in increments of ±720° to a closed toroidal loop. The same applies to single-sided, non-orientable Möbius bands, and thus there are two regular homotopy classes for immersed Möbius bands, and they differ in the amount of "built-in" twist by exactly 360°. All the shapes depicted in Figure 1 belong into the same regular homotopy class (**ML**) with a built-in left-turning twist of 180°. This twist should be measured when the sweep path forms a planar circle; if the 3D shape of the sweep is changed, the "apparent" twist as compared to a rotation-minimizing sweep may also change. Mirroring, on the other hand, would turn the left-turning Möbius band (**ML**) of Figure 1a into a right-turning one (**MR**), and thus cast it into a different regular homotopy class.

Several of the stages of different apparent twistedness depicted in Figure 1 have been exploited by various artists: The almost circular shape (Fig.1a) can be found in a wedding band [21] (Fig.2a). The shape shown in Figure 1b has been celebrated in a sculpture by Max Bill [1] (Fig.2b), and the configurations (1c) and (1d) appear in drawings by M.C. Escher [8][6] (Figs.2c and 2d).



(a) (b) (c) (d)

Figure 2: *Artistic Möbius bands: (a) wedding band, (b) sculpture by Bill (from [1] used by permission), (c, d) drawings by Escher (from [8],[6] used by permission).*

If we focus on the edge of the circular Möbius band (Fig.1a) and include a tiny sliver of the band's surface (a neighborhood) with it, then this edge-band forms a double loop with an apparent twist of 360°. This double loop can be unfolded into a figure-8 shape without changing the apparent twist as explained in Figure 18 in [18]. This unfolding of the edge will stretch the narrow Möbius band into an extended surface that resembles a two-pouch basket (Fig.3a), but which topologically is still a Möbius band. Further deformations can be applied to this edge band, creating different kinds of "baskets" or "goblets." In particular, we can further un-warp the Möbius band edge into a circle. This regular homotopy will change the edge band's apparent twist by ±360°. If we let the twist add up to 720° we obtain the Sudanese Möbius band [12] depicted in Figure 3b. Alternately we can let the twist cancel out to zero and then obtain another interesting warped surface – which happens to be equivalent to a Boy surface [2][3] minus a disk (Fig.3c). Some of these shapes play an important role in the analysis of various Klein bottles.

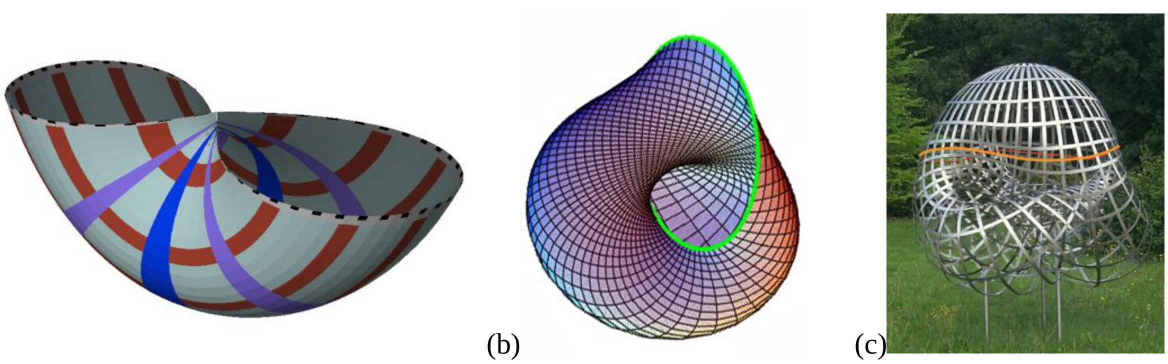
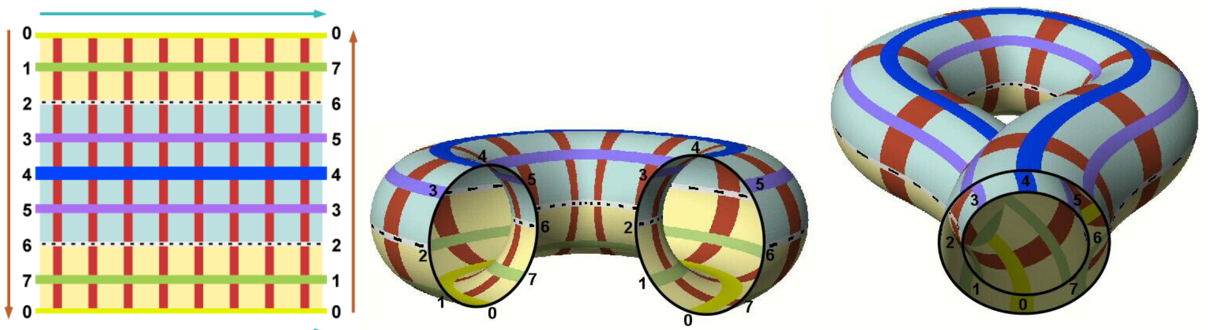

(a) (b) (c)

Figure 3: *Deforming the Möbius band edge: (a) into a figure-8 shape with an apparent twist of 360°; or into a circular loop: (b) with 720°of apparent twist, or (c) with zero apparent twist, when the Boy surface is cut at the red line (photo from the Archives of the Mathematisches Forschungsinstitut Oberwolfach, used by permission).*

3. The Classical Klein Bottle

We begin our investigation of the realm of Klein bottles with a formal construction of the well-known "inverted sock" shape. We start with the parameterized, decorated rectangular domain depicted in Figure 4a. As for the case of torus construction, we first merge the two (horizontal) edges marked with parallel cyan arrows to form a generalized cylinder. For now we give this tube simply a round or oval profile (**O**) (Fig.4b). We then close this tube into a **J**-shaped loop (Fig.4c), so that its two ends can be joined with the reversed orientation indicated by the labeling and by the two anti-parallel brown arrows in Figure 4a.

Since we are now dealing with single-sided surfaces, I will use a somewhat different coloring scheme from the one I had used for tori. While I maintain the color **red** for the *meridian* bands, I split the domain into two halves with different background colors and give their center bands a fully saturated color, either b**l**ue or ye**ll**ow, to mark the *parallel* parameter lines; these two regions will always form the two Möbius bands into which a Klein bottle can be partitioned. Parallel to these central bands I draw lines of less saturated color (o**l**ive, purp**l**e) that will wrap twice around the Klein-bottle loop, thereby executing an even number of 180° flips. Thus a key difference to *Tori Story* is that not all parallels are the same anymore. Because of the reverse labeling along the two vertical edges of the rectangular domain, there are only exactly two parallels that meet up with themselves; those form the center-lines of two Möbius bands with ±180° twist. All other parallel parameter lines form double loops with a ±360° twist.



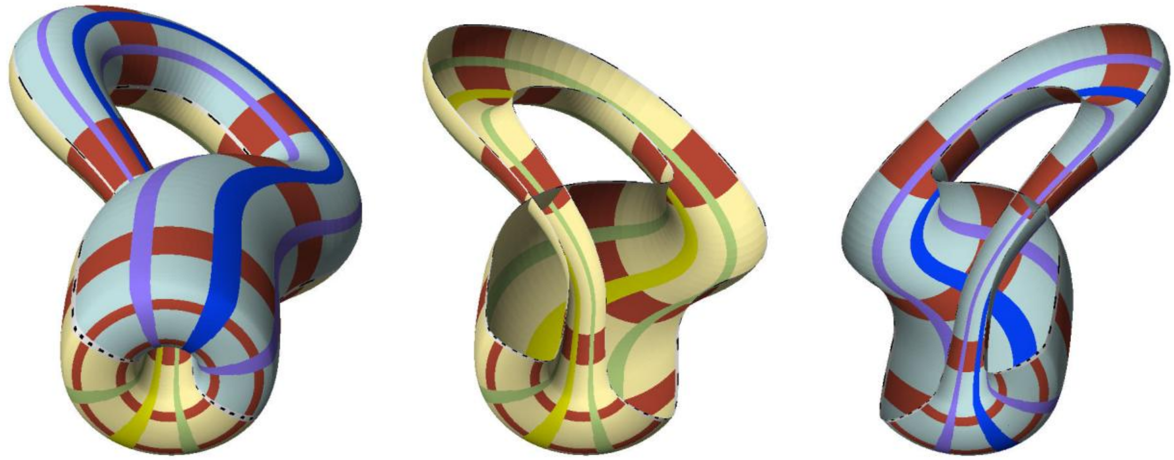
(a) (b) (c)

Figure 4: *Formal construction of a Klein bottle: (a) its rectangular domain; (b) the domain rolled up into a tube; (c) the tube bent into a **J**-formation with its two ends lined up in a concentric manner.*

For the tube with the **O**-profile, the closing of the loop is most conveniently done by narrowing one end and inserting it sideways into the larger end of the tube – forming a **J**-shaped loop (Fig.4c); this configuration properly lines up all the numbered labels. Now the two concentric ends are merged by turning the smaller one inside out. This yields the classical "inverted sock" Klein bottle, named **KOJ** (Fig.5a). In this case, both "special" parallels form two Möbius bands of opposite handedness. Correspondingly, the **KOJ** Klein bottle can be partitioned into two Möbius bands (yellow and blue) having opposite handedness as shown in Figures 5b and 5c. We can denote this symbolically as:

MR + **ML** = **KOJ**.

The *Tori Story* paper [17] spent much effort to analyze the amount of twist built into the toroidal ring, because tori that differ in their amount of built-in twist by 360° belong to different regular homotopy classes. Interestingly, for the classical Klein bottle twist is a non-issue! By rotating the handle around the symmetry axis of the "inverted-sock" turn-back, any amount of twist can be added or removed [14]. This is a consequence of the fact that in Figure 4a the labeling at the left and right sides of the fundamental rectangular domain can be shifted cyclicly by any arbitrary amount; but there will always be exactly two labels that are lying on the same *parallel* parameter lines. These two parameter lines then form the center-lines of the two Möbius bands of opposite handedness into which this Klein bottle can be decomposed.

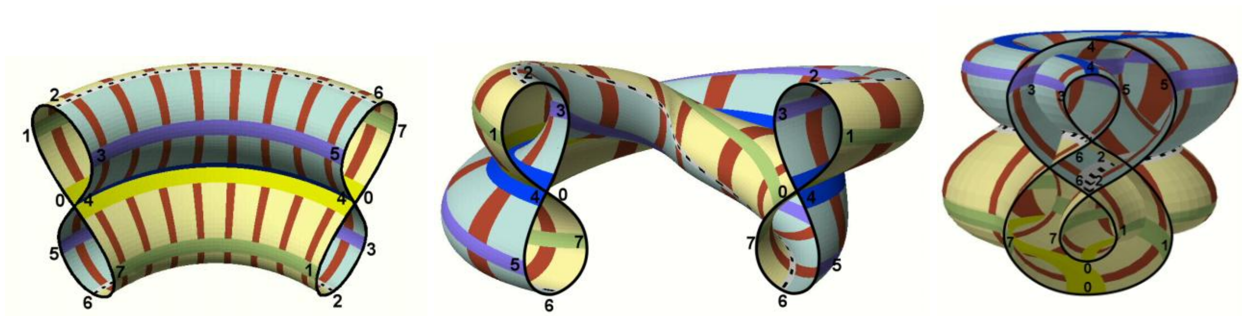


(a) (b) (c)

Figure 5: *The ordinary "inverted sock" Klein bottle resulting from the construction in Figure 4: (a) the complete Klein bottle **KOJ**; (b) the lower half of it is a right-handed Möbius band (**MR**); (c) the upper, left-handed Möbius band (**ML**) shown flipped over.*

4. Figure-8 Klein Bottles

When merging the two parallel edges marked by cyan arrows in Figure 4a to form an initial tube, we are not forced to form a round, circular **O**-profile. Instead we may form a figure-**8** cross section (Fig.6a) or an even more complicated, multiply-rolled generalized cylinder, as was discussed for tori [17]. This results in various Klein bottles that may belong to different regular homotopy classes, and which thus cannot be smoothly deformed into the classical "inverted-sock" shape.



(a) (b) (c)

Figure 6: *Constructions for figure-8 Klein bottles: (a) figure-8 tube; (b) the tube twisted through 180° so that the two ends can be merged into a toroidal loop; (c) a new way to line-up the number labels.*

For the tube with the figure-**8** profile, there are a few different ways in which we can fuse the tube ends with the required reversed orientation, and they will result in different Klein bottles. For instance, we can bend the tube into a simple toroidal loop (an **O**-shaped path) and give the figure-**8** cross-section a 180° torsional flip (Fig.6b). This flip can either be clockwise (right-handed: **R**) or counter-clockwise (left-handed: **L**) and this will result in two figure-**8** Klein bottles that belong to two different regular homotopy classes – for the same reason that the Möbius band shown in Figure 1 cannot be smoothly transformed into its own mirror image. We call the resulting two Klein-bottle classes **K8R-O** (Fig.7) and **K8L-O** (Fig.8), respectively. For both classes the two Möbius bands that form the Klein bottle, and which happen to intersect along their center lines, are also shown separately in Figures 7 and 8 (a, b).

(a) (b) (c)

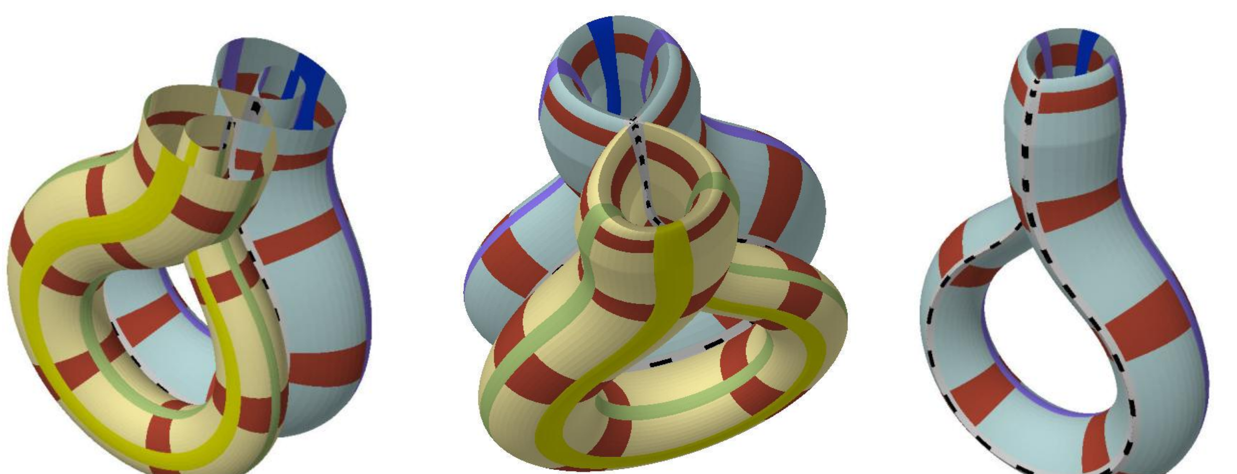
Figure 7: *Two right-handed Möbius bands **MR** (a,b) form a right-handed Klein bottle of type **K8R-O** (c).*

(a) (b) (c)

Figure 8: *Two left-handed Möbius bands **ML** (a,b) form a left-handed figure-**8** Klein bottle **K8L-O** (c).*

5. Yet Another Type of a Klein Bottle

But there is even a third way in which the **8-**profile tube can be closed into a Klein bottle (Fig.9):



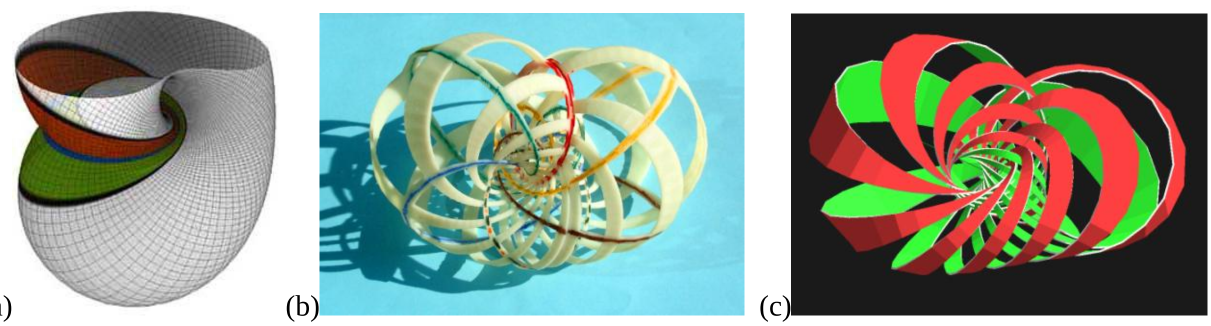
(a) (b) (c)

Figure 9: *A Klein-bottle based on a "inverted double sock": (a) without end-caps to show the nesting of the tube profiles; (b) completed Klein bottle of type **K8L-J**; (c) one of its two **ML** Möbius bands.*

This new type of Klein bottle is constructed by forming the same kind of **J**-shaped sweep path as for the classical Klein bottle and fusing the two nested figure-**8** profiles by turning one of them inside out. To do this smoothly, we use an asymmetrical figure-**8** profile in which one lobe is larger than the other one. As we sweep from one end of the tube to the other one, the larger lobe shrinks and the smaller one grows, so that the end-profiles can be nested as shown in Figures 6c and 9a. Now a nicely rounded, **8**-shaped end cap can smoothly close off this Klein bottle (Fig.9b). I have not seen this particular Klein bottle depicted previously. It is rather special, since it does not just have a single self-intersection line like all the other models, but features two triple points. These triple points occur where the self-intersection line of the figure-**8** cross section passes through the wall of the other tube near the mouth of the Klein bottle. If we split this Klein bottle into its two constituent Möbius bands, they can be separated cleanly along the closed self-intersection loop passing through both triple points. Each component on its own resembles an "inverted sock" Klein bottle shape with a sharp crease line, which itself exhibits a Möbius twist (Fig.9c).

6. Minimum Energy Klein Bottle

Since it is not clear which one of many possible shapes within the same regular homotopy class should be singled out as the generic representative, we may look for an objective measure for defining the "best" shape. The shape with a minimal amount of total bending energy may be used in this context. This measure, which integrates mean curvature squared over the whole surface, is scale-independent and thus well-suited for this purpose. Lawson [11] has defined the minimum energy forms for many topological shapes. Kusner [10] has conjectured that this Willmore energy [22] is indeed minimized by a Klein bottle described by Lawson. Even though nicely depicted by Polthier [13], this configuration is the most difficult one to understand of all the Klein-bottle shapes discussed so far.



(a) (b) (c)

Figure 10: *Lawson minimum-energy Klein bottle: (a) Shell with center portions of Möbius bands [13] (top cut off); (b) an FDM model in which the parallel parameter lines have been hand-colored; (c) the set of circular, but twisted (by 360°) meridians.*

7. Regular Homotopy Classes

Now that we have examined several different construction schemes for making Klein bottles, we can try to group them into the expected number of regular homotopy classes, and then single out a representative for each class that describes this class in the most understandable way.

Regular homotopy transformations cannot perform mirroring operations; they cannot turn a left-twisting Möbius band into a right-twisting one. Thus we can immediately identify three different classes represented by: **K8L-O** = **ML**+**ML**, **K8R-O** = **MR**+**MR**, and by: **KOJ** = **ML**+**MR**; the fist two are chiral, and **KOJ** is amphicheiral. These three classes are <u>structurally</u> different; they do not depend on any markings of the surface. This exposes yet another difference to the world of tori [17], where there are only <u>two</u> structural classes: one formed by the tori **TOO**, **TO8**, and **T8O**, and the other one by **T88** by itself.

Now, where do we find the 4th expected type of Klein bottle? It has to be composed of two Möbius bands of the same twistedness; thus structurally it belongs into the same class as **KOJ**, and it can only be distinguished from the classical **KOJ** if we place some markings (e.g., a coordinate grid) on its surface.

Could the Lawson Klein bottle discussed above be the sought-after 4th representative? The Lawson surface (Fig.10) has a single circular intersection line, and this line happens to represent the coinciding center lines of two Möbius bands with the same handedness. Indeed the Lawson Klein bottle does come in two different chiral versions [10]. So we can conclude that they must correspond to the two classes: **K8L-Lawson** = **K8L-O** and **K8R-Lawson** = **K8R-O**. Similar argumentation also rules out the "double-sock" Klein bottle (Fig.9). It has C_2 rotational symmetry and is composed of two Möbius bands of the **same** type. There are again two mirror-image versions: **K8R-J** = **K8L-O** and **K8R-J** = **K8R-O**.

In the world of tori, drawing an explicit parameter grid allows us to introduce Dehn twists [6] in the meridial direction (M-twist) or in the longitudinal ("equatorial") direction (E-twist). The introduction of either one of such twists yields a torus that belongs into a different regular homotopy class for marked surfaces. An M-twist of 360° takes **TOO** into **TO8**, and an E-twist of 360° takes **TOO** into **T8O**. Thus we might also want to try this operation on **KOJ**. However, in the world of Klein bottles the introduction of Dehn twist leads to inconsistencies in the resulting texture mapping [19]; so this approach is futile.

Another operation that changes the regular homotopy class of a torus is adding a collar as illustrated in Figure 15 in [18]; thus we give this approach a try. In Figure 11a a new texture is introduced with longitudinal directionality; this gives the classical KOJ a pattern in which the arrows are coming out of the mouth (Fig.11b). Now we add a self-intersecting collar around the rim of the mouth (Fig.11c). But this collar can easily be eliminated by inflating the blue/purple part of the tube. The net effect is that the roles of the thin and the thick branches have been reversed, and the directional texture arrows now are pointing into the mouth (Fig.11d). So far, this is the best representation I can offer for the 4th class.

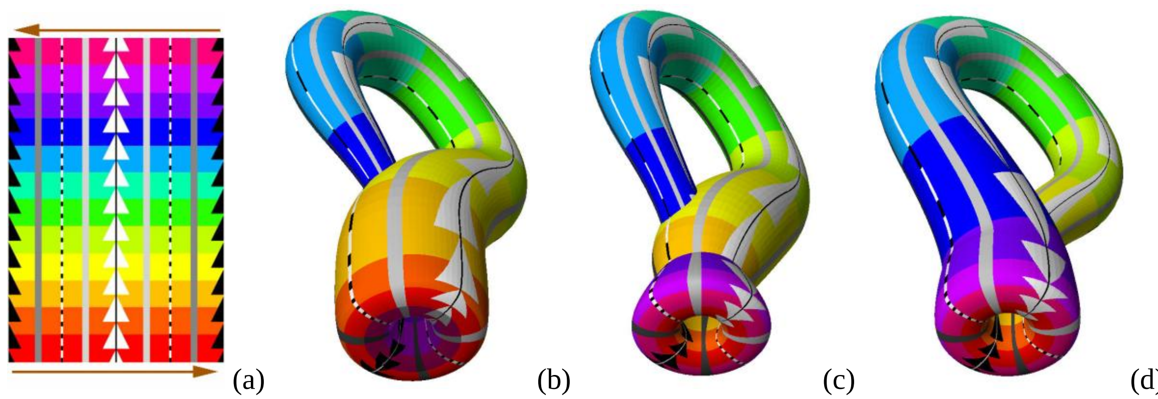
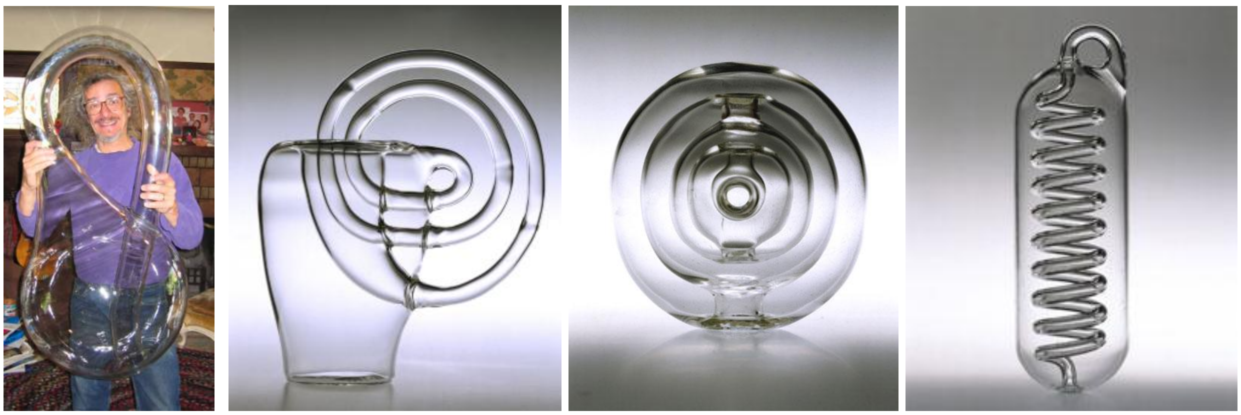


Figure 11: *(a) A new texture with longitudinal directionality; (b) classical **KOJ** with this texture applied; (c) **KOJ** with a collar added at the mouth; (d) **KOJ** with inflated blue/purple branch.*

When faced with the fancy and perhaps confusing contraptions presented in the next two sections, how can we figure out to which regular homotopy class a particular surface belongs? Since these objects have no decorations, we only need to figure out into which structural class they belong. First we make sure that the surface is indeed single-sided and has Euler characteristic EC = $V - E + F = 0$; i.e., that it is essentially a single, contorted, self-intersecting "tube", possibly with some "inverted sock" turn-backs – but with no branching. Then we start drawing a set of three "parallel" lanes and wind this "highway" over the surface until it joins itself again. If each of the three lanes joins itself and exhibits a twist that is an even multiple of 180°, then we have found a set of meridians, and, based on whether it is twisted or not, we can readily determine whether the surface is in one of the **K8R/L** classes or in **KOJ**. If only the center lane joins itself, and the other two lanes form a double-loop over the surface, then we have found one of the two Möbius bands. The portion of the surface not yet covered must then form the other Möbius band. We need to determine the twist of both of them! To find a representative meridian strip, we start a second "3-lane highway" roughly perpendicular to the first one and look for a way to let it close on itself after only a single intersection with each of the two Möbius bands. The twistedness of these three bands uniquely characterizes the structural regular homotopy class of the shape in question.

8. Fancy Klein Bottles

Cliff Stoll offers a wide variety of Klein bottles for sale [20]. Some of them are very large (Fig.12a). Even more elaborate glass models have been constructed by Alan Bennet for the Science Museum in South Kensington, UK [4]. These range from an "inverted sock" type with a multiply looped (Fig.12b) or helically twisted (Fig.12d) handle to contraptions that nest several such shapes inside one another (Fig.13a). The latter are topologically no longer simple Klein bottles, since they may be of higher genus or they may form multiple, individual, but interpenetrating surfaces. The examples (Fig.13) are shown here mostly for the enjoyment of the reader, but they may also be used as study objects to train the reader's skills in determining the regular homotopy class of a particular glass sculpture.



(a) (b) (c) (d)

Figure 12: *Fancy Klein Bottles of type* ***KOJ****: (a) Cliff Stoll with a very large classical Klein bottle (from [20] used by permission); (b, c, d) Klein bottles by Alan Bennet exhibited at the Science Museum / SSPL in South Kensington, UK (from [4] used by permission).*

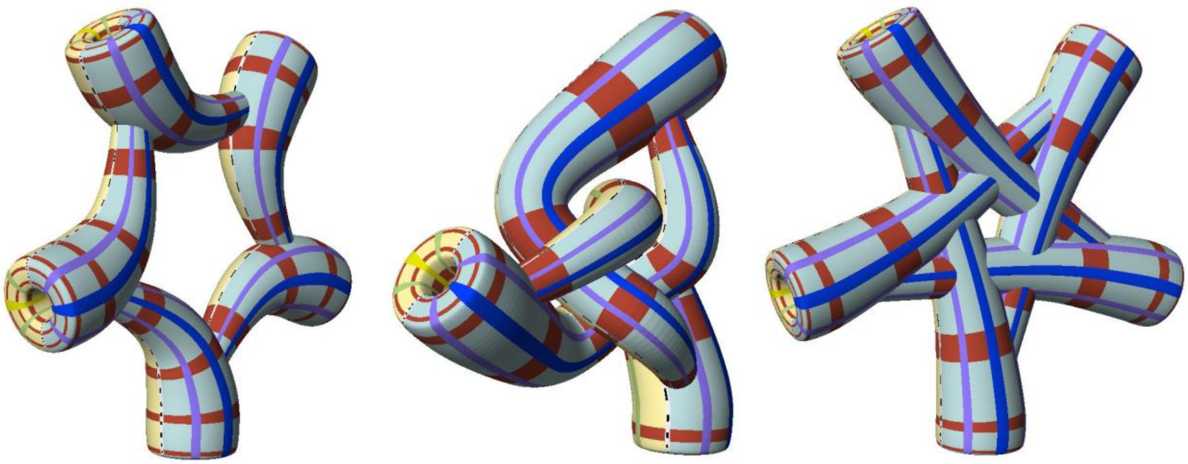
(a) (b) (c) (d)

Figure 13: *Beyond ordinary Klein Bottles: (a) three nested Klein bottles; (b) 2-handle Klein bottle; (c) 3-handle Klein bottle; (d) three merged Klein bottles forming a non-orientable surface of genus 6. All glass models are by A. Bennet; they are located at the Science Museum / SSPL in South Kensington, UK (from [4] used by permission).*

9. Klein Knottles

Inspired by the creative bottle shapes shown above, I searched for other intriguing geometries that topologically are proper Klein bottles. Given my investigation of knotted shapes in the past [15] [16], it was natural for me to look for knotted varieties of Klein bottles; I call these geometries "Klein Knottles." A simple way to enhance the visual complexity of a Klein bottle is to put more than one – but definitely an odd number – of "inverted sock" turn-backs in series (Fig.14a). All of these surfaces belong in class

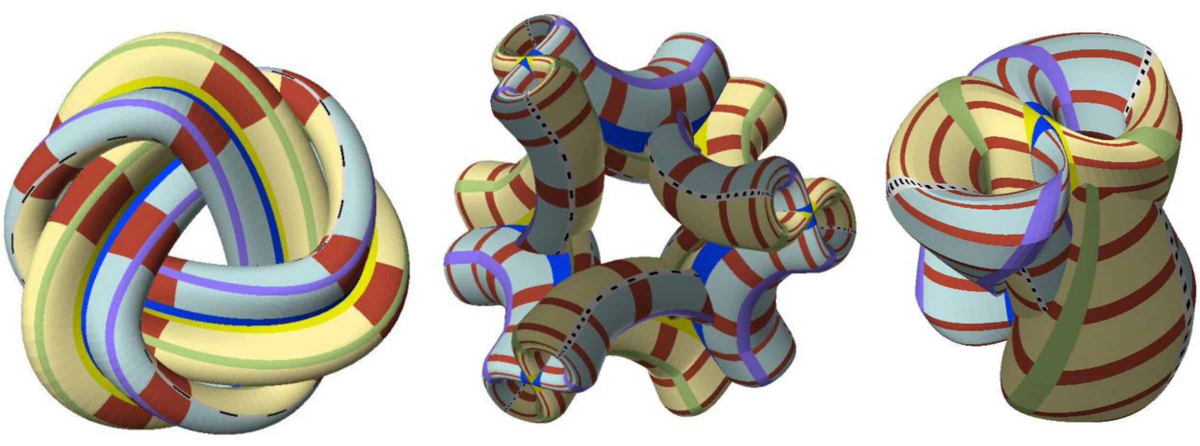
KOJ; pairs of subsequent turn-backs can always be created or eliminated by doing an inversion of the segment in between using the Cheritat eversion move [5], which is also illustrated in Figure 5 in [17]. Any chain of such turn-backs can readily be deformed into a knotted geometry, such as the simple trefoil shown in Figure 14b, since tube branches are allowed to slide through one another. Of course, any intermediate state in such a knotting process may also be the final target of an artistic representation of a Klein bottle, since there is no limit on the number of self-intersections allowed (Fig.14c).



(a) (b) (c)

Figure 14: *Examples of Klein Knottles: (a) ring of five **KOJ**; (b) three **KOJ** forming a trefoil knot; (c) five **KOJ** forming an interpenetrating Knottle.*

We can also form Knottles with figure-**8** profiles. The toroidal Klein bottles of type **K8R-O** and **K8L-O** can readily be deformed into any knot desired (Fig.15a) – just as long as we make sure that the cross section makes an odd number of 180° flips overall, so that we obtain the inside/outside switch-over needed to make a Klein bottle. We can also form chains with multiple figure-**8** turn-backs; each one of them acts as an inside/outside switchover. We can even make Klein bottles with an even number of such turn-backs, because we can always insert an extra 180° of twist in the figure-**8** profile. Figure 15b gives an example with six turn-backs – each curved segment also effectively adds 90° of twist into the chain. Figure 15c shows a minimal configuration of this kind with only two turn-backs and with 90° of twist in each leg.



(a) (b) (c)

Figure 15: *Klein Knottles with figure-**8** cross sections: (a) **K8R-O** wound into a trefoil knot; (b) six **K8J** turn-backs with suitable twist; (c) two **K8J** turn-backs with 90° twist each.*

10. Conclusions

Klein bottles are fascinating geometric objects. Most people are familiar with only the "inverted sock" type (**KOJ**). Many mathematical texts also refer to "the other" Klein bottle – the twisted figure-**8** shape – which actually comes in two different chiral versions (**K8L-O**, **K8R-O**). Only a few papers or web pages also present Lawson's minimum-energy Klein bottle, which also comes in two chiral forms that belong to the same two regular homotopy classes. But there is also the possibility of forming turn-backs with a figure-**8** profile and I have not seen this one depicted before (Fig.9).

Of course, there are infinitely many possible geometries that constitute Klein bottles, i.e., single-sided surfaces with Euler characteristic zero. Even with marked surfaces, each of these surfaces can be smoothly deformed into one of the four representatives presented in this paper. Finding the most elegant transformation that will actually accomplish this reduction is a much harder task; it deserves more study [19]. In the meantime the reader may simply enjoy the beautiful glass models created by Alan Bennet and Cliff Stoll.

Acknowledgements

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