Imagining Negative-Dimensional Space

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Abstract

The goal of this workshop is to induce the experience of contemplating negative-dimensional space. The authors are developing a performance art piece about negative-dimensional space, to be performed in Berlin in 2013. Workshop participants will preview and test out various thought experiments, movement-based lessons, and intuition-explorations, all aimed at an experience of negative dimensions. The workshop, accessible to all, will also serve as a demonstration of the performance/lecture genre of performance art.

Introduction

By the 1940s, topologists had developed a fairly thorough basic theory of topological spaces of positive dimension. Motivated by computations, and to some extent aesthetics, many topologists began searching for mathematical frameworks that extended our notion of space to allow for negative dimensions. It wasn't until the 1960s that one was constructed – the *category of spectra*. A *spectrum* is a generalization of space that allows for negative dimensions. The study of spectra, called stable homotopy theory, is a robust and elegant field.

To understand spectra rigorously, of course, would require several graduate-level courses in mathematics [2,5]. In our workshop, we aim to induce the experience of contemplating spectra, and "negative-dimensional space," in a way that is accessible to the layperson. To do this, we turn to holistic, pre-rational, and post-rational understandings, and the performance/lecture.

Performance/lecture is a type of performance art that balances the outward-facing conceptual exploration of a performance with the inward-facing conclusions of a lecture. Growing out of conceptual art of the 1950s, and the Happenings of the 1960s, performance art is now one of the main threads of contemporary art. By using a performance/lecture to move the participants closer to an experience of negative-dimensional space, we will be presenting a synthesis of the most contemporary mathematics and the most contemporary art.

Mathematical Background

A *topological space* X is a set of points with a well-defined notion of nearness (called *the topology on X*) [9]. An n-*dimensional space* is a topological space that has a consistent dimension n, in the sense that if we zoom in on an arbitrary point p, the points in X near p look like n-dimensional Euclidean space. So, for example, the circle S^1 is one-dimensional, because nearby each point it looks like a piece of the 1-dimensional real line $\mathbb{R}^1$. The surface of a sphere S^2 is a two-dimensional, because zooming in at the surface looks like a piece of a 2-dimensional plane $\mathbb{R}^2$.

Topologists are interested in the global properties of spaces, for example: how many "holes" does it have of different dimensions? Because of this focus, stretching and squishing is allowed. A square is topologically equivalent to a circle, and the shape of the letter "Q" is equivalent to "D," but not to "B" or "L."

One of the most important and useful operations on spaces is called *suspension*, denoted Σ. We won't give the definition here, but simply illustrate the suspension of two points, and the suspension of a circle[1]:

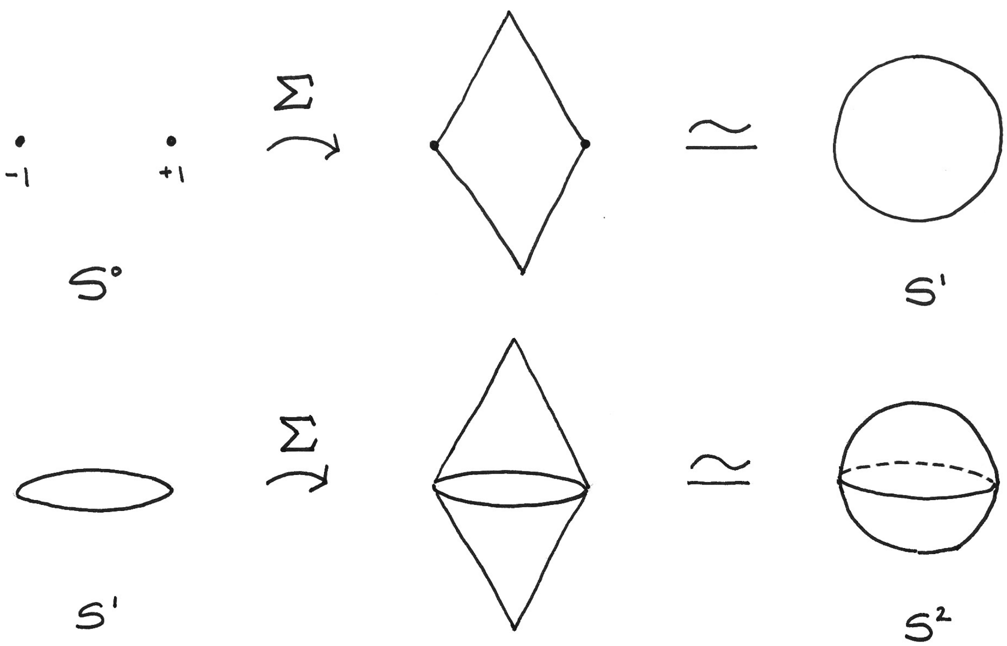


Figure 1: *The suspension of S^0 is S^1. The suspension of S^1 is S^2.*

Just as the one-dimensional sphere S^1, the circle, is defined to be the set of points in $\mathbb{R}^2$ that are a distance of one from the origin, the zero-dimensional sphere S^0 is all the points in $\mathbb{R}^1$ that are a distance of one from the origin. Therefore S^0 is simply two points, $\{-1, +1\}$. The above illustration shows that the suspension ΣS^0 of S^0 is topologically equivalent to S^1, and similarly ΣS^1 is topologically equivalent to the two-sphere S^2.

In general, given an n-dimensional space X, the suspension ΣX will have dimension $n+1$. In particular, $\Sigma S^n \simeq S^{n+1}$. Thus suspension gives a way of moving up in dimension, like adding one rock to a pile of rocks.

[1] Illustrations by Shannon Wallace (greenleavesblueprints.com)

For various reasons[2], mathematicians in the 1950s decided it would be nice to be able to have an inverse operation Σ^{-1}, called *desuspension* [10]. But then what happens when you desuspend the zero-dimensional sphere S^0? Clearly, we should get S^{-1} but what does that look like?

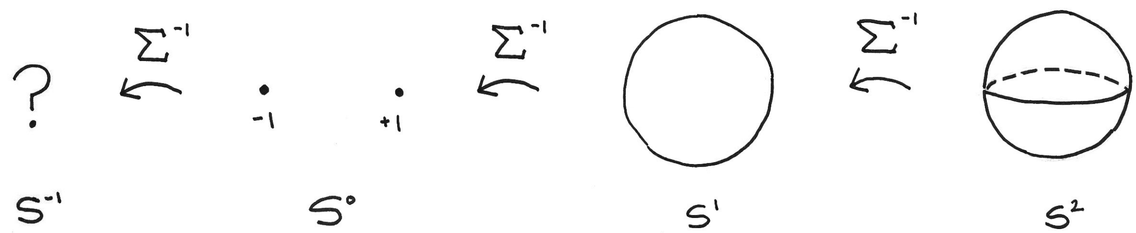


Figure 2: *Desuspension: what does it lead to?*

This is just one way to watch our understanding of space break down as we try to pass to lower dimensions. In the workshop, we will work through several others, some of which are illustrated below.

The same paradox arises from repeated subtraction of natural numbers. Given a pile of rocks, we can take away rocks only until all the rocks are gone. If we want to continue subtracting, we need to break free of our conception "numbers = piles of rocks," to find a new metaphor. The invention of negative numbers was a revolution because it declared "numbers = points on a line." To this day, most of our reasoning about negative numbers makes use of the image of the number line [8].

What would it look like, what would it *feel* like, to break free from our usual notions of "space" and "dimension" to a new realm, where negative-dimensional space makes sense? This revolution has already taken place in topology, but to date the story has not been made accessible to the public.

Art Background

For the past century, intersecting with advances in Modernism, formalism, and material exploration, there has been a fundamental shift in the material paradigm of art. First the invention of photography forced us to reconsider the meaning of image-making itself, shifting the emphasis towards the act of making and the non-pictorial function of material, a.k.a. Modernism. Then, with advent of more streamlined mass production in the 1950s and 1960s, the importance of materials themselves came under question, and Western art was philosophically launched into Phenomenology, post-structuralism, and a (then) new notion of idea-as-material, a.k.a. Conceptual art.

Thus, we now have a long-established operating system wherein artists approach works with an immaterial theoretical structure, and any material qualities that do emerge do so out of formal necessity, but otherwise are not essential to the "existence" of the piece. Formality (that is, that which pertains to Form) is not a prerequisite for art. Substance is defined as something altogether different.

Another movement that developed out of Conceptual art was the work of Happenings in the 1960s, which over subsequent generations has transmuted into various forms of performance art. Much of performance art is not explicitly theatrical with clear-cut subject-object terms, but is socially "relational."

[2] For one, desuspension would make the category of spaces a triangulated category. Second, if arbitrary coproducts were allowed, desuspension would result in all cohomology functors being representable.

Relational art is concisely defined as "a set of artistic practices which take as their theoretical and practical point of departure the whole of human relations and their social context, rather than an independent and private space" [3].

Therefore, the act of performance can manifest in many more ambiguous forms than that which is tableau-oriented, such as public intervention, public interruption, invisible actions, symposiums, lectures, workshops - all producing intersubjective encounters, disbanding conventional producer-audience roles, toppling objectness. But while these actions exist in ambiguous forms, let it be clear that the conceptual intentionality is highly precise. The purpose is not to behold a discrete object, but rather to re-examine the subject, reconsider ourselves.

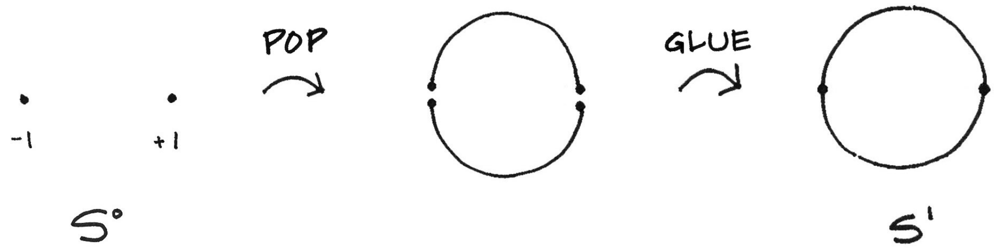
Workshop Content:

In an engaging and participatory atmosphere, and in a way that is accessible to all, we will:

(1) Explore common notions of space and dimension, and, using thought and movement, experience their limitations.
(2) Learn about the generalization of space, to spectra, that has allowed topologists to work in negative dimensions.
(3) Through activities and historical examples, induce and reflect on the experience of using one's own minds to transcend one's habits and modes of thought [7].
(4) Underscore the problematic authority of esoteric knowledge, as it pertains to certainty and the role of mathematics in society [6].
(5) Through performance, present a time-based mathematics full of ambiguity, aesthetics, and paradox, blossoming in tandem with passive and active rationality [4].
(6) Contextualize this art-without-object within relational art [1].
(7) Demonstrate performance/lecture as a pedagogical alternative or complement to conventional learning environments.

Thought Experiment Examples:

One workshop activity will challenge participants to push their understanding of dimension and space, using various thought experiments. Here we illustrate some examples. The first two show how we can visualize moving up in dimension, from zero to one, and one to two. Can you imagine continuing up to three and four?



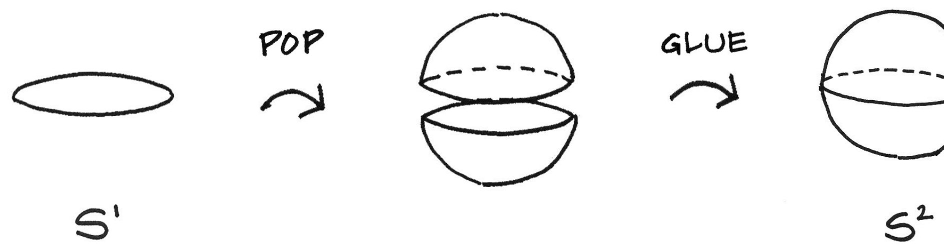


Figure 3: *Going up in dimension by "popping out and gluing."*

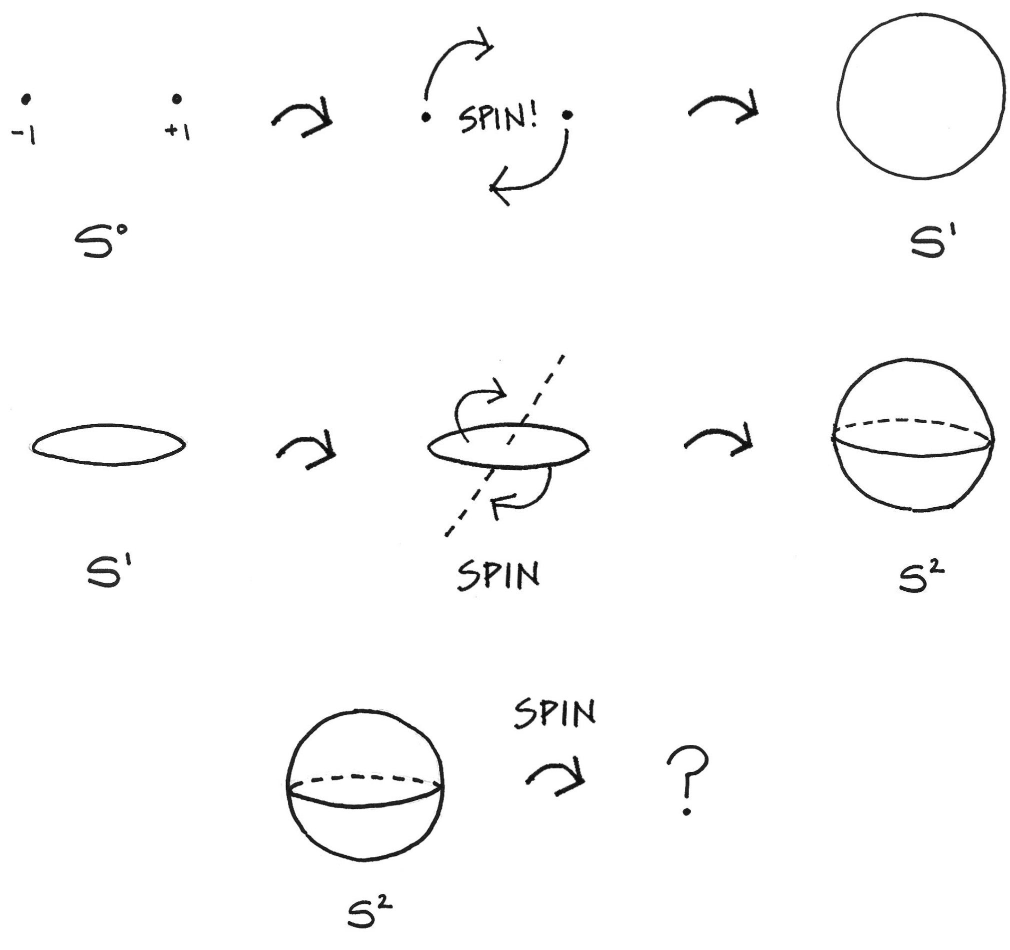


Figure 4: *Going up in dimension by "spinning."*

The final illustration suggests how we might visualize moving down in dimensions, from two to one, and one to zero. What comes next?

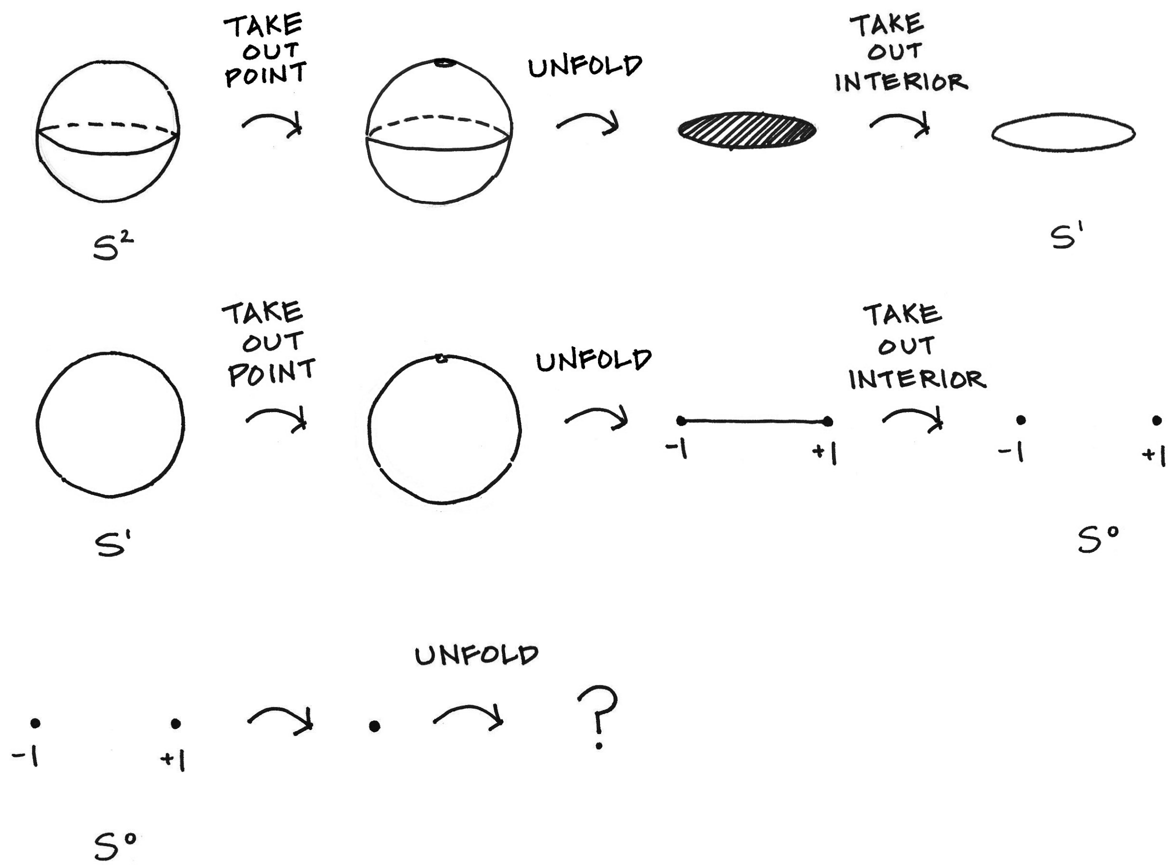


Figure 5: *Going down by "poking, unfolding, and carving."*

References

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