Images and Illusions from Orthogonal Pairs of Ellipses

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Abstract

In this paper I create planar images using lists of pairs of orthogonal ellipses with a common center (POEs) as the single graphic element. Changing attributes of the POEs such as size, color, position, rotational orientation, and excentricity using linear and trigonometric functions gives rise to a wide spectrum of images. Some combinations of parameters appear to produce abstractions of three-dimensional organic creatures and of phantastic assemblages that might have come out of the Burgess Shale in Alberta, Canada.

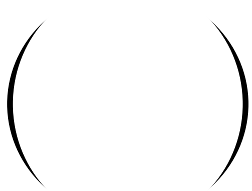
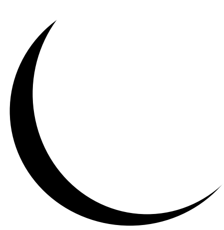
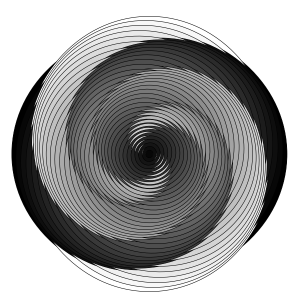
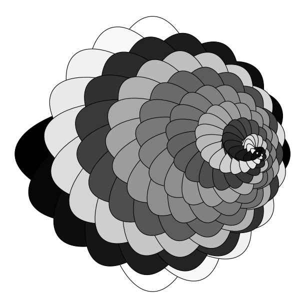
Introduction

One half sphere of the moon is illuminated by the sun so that the boundary of the shadow is a great circle of the moon. We see half of that great circle move across the face of the moon as the relative positions of sun, earth and moon change during one orbit of the moon around the earth. When we take a great circle and project it on the plane we obtain a general ellipse; there are two extremes: a line segment representing the diameter and a circle representing the equator. Planetary bodies such as the earth and the moon are shaped like elipsoids with varying degrees of flattening, e.g., 0.001248 for the moon and 0.003353 for the earth with respect to equatorial and polar radii. Therefore, drawing a moon sickle amounts to drawing pieces of two ellipses. This simple reality of nature is the basis for the images that I want to show you. I render all graphics in *Mathematica* 8 and use only two-dimensional objects: ellipses.

It all started with two ellipses with semi-axes of 105 and 100 units, respectively; that is ellipses with excentricity $\epsilon = \sqrt{(a^2 - b^2)/a^2}$ of about 0.3049 and a geodesic flattening $f = \frac{a-b}{a}$ of about 0.04762. When I switch the major and minor axes I obtain an orthogonal ellipse, and when I draw a white ellipse on top of a black ellipse I obtain a pair of symmetrically placed black sickles: ( ). Such a figure consisting of a pair of ellipses with the same center and with switched semi-axes I call a **pair of orthogonal ellipses** or a **POE** for short. Shifting the center of the white ellipse along the diagonal in the first quadrant produces a single sickle like that of a waning moon. Abstracting from the stark black and white image I color the two ellipses, though the graphics in this paper are rendered in gray scale.

Spirals and Shells

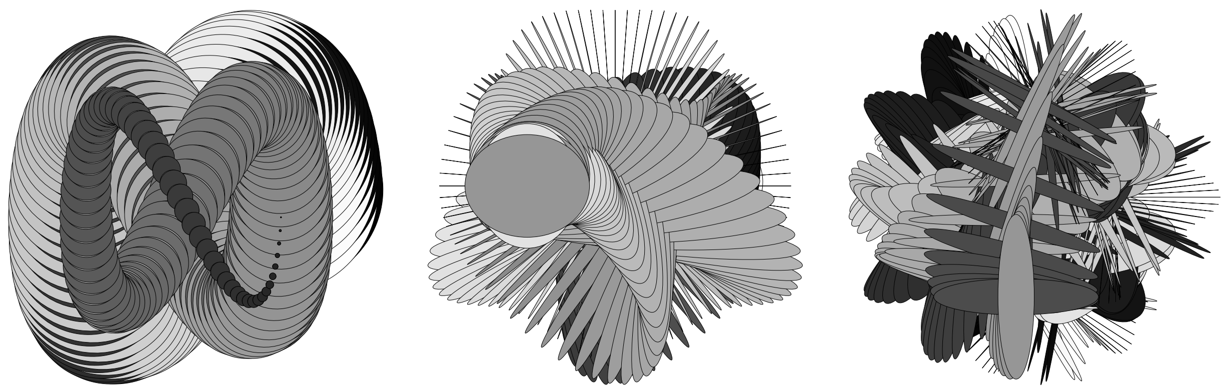
One obvious next step is to use layers of POEs in order to create patterns that are mathematically interesting and to render complex images that are based on this one single visual element. A common mathematical method to change or move figures is to use linear or affine transformations and scaling operations. First, I create lists of POEs from a single POE that I rotate in discrete steps around their common center, synchronously shrink their size as well as change their color attributes. All modifications are implemented with linear functions. The paired sickles rather than the ellipse shapes dominate the images and form, predictably, spirals. When next I move the common center of the POEs far enough along a straight line, then the flat, matched sickles of the POEs are moved to a point outside the original POE. Our eyes frequently see those figures in three dimensions rather than as planar drawings. They easily are visually interpreted as top portions of sea or snail shells or of cones or petals.

Worms and Phantastica

The two spiral and shell images above are quite regular and predictable despite the variety that they exhibit. One reason is that all attributes of the individual POEs change linearly. As a next level of complexity I move the center of the POEs along some open or closed curve, for example, an exponential curve or a Lissajous curve or a cycloid or some parametric curve defined by a polar function such as $r(\phi) = \sin^3(3\ \phi) + \cos(4\ \phi)$.

Many of these planar curves exhibit multiple symmetries and the parametric trigonometric curves have self-intersecting patterns that depend on their periods and can be quite intricate. However, when they are overlaid in discrete steps with POEs of various sizes, color combinations, and semi-axes modified by functions such as f(t) = $(\cos^2 t,\ \sin^2 t)$ the resulting images hide the underlying curves and appear three-dimensional. In many cases they form outright fantastic shapes that belie the simplicity of the underlying two-dimensional geometric objects: curves and ellipses. Here are three representative examples.



There do not appear to be any limitations that ellipses and planar curves impose on the imagination. Depending on the sizes and total number of the POEs the curve that their center follows may be apparent as in the first image, $g(t) = (\cos t, \sin 2t)$, or totally obscured by the density of the POEs as in the third image, $h(t) = \left(\sin^{10}(10\,t)\ +\ \cos(10\,t)\right) \times (\cos t, \sin t)$.

Brief Notes on Implementation

As I mentioned in the beginning, I created all images in *Mathematica*, but I am not displaying the detailed definition of the function that creates the images. However, the function invocation for the second of the three images in the previous section gives the flavor of its design:

```
Graphics[rotateEllipsePair[butterfly[#,{3,4}]&, f, {2.0,1.0}, {{0,Pi},{-1,1},{-Pi,Pi},90},
        colorShift → {{1,1,-1},{1,-1,1}}, edgeForm → 0.0005, colorRendering → False]]
```

Here argument `butterfly[#,{3,4}]&` is the parametrized version of the function $r(\phi)$ and the second argument f is the function f(t), both mentioned in the previous section, which control the underlying path and the excentricities of the two orthogonal ellipses in a POE, respectively. The third argument gives the base values for the two semi-axes. The three ranges specify the extent of the (parametric) arguments for the curve, the semi-axes and the rotation of the POEs. This particular image consists of 90 POEs. Finally, the three options modify the linear colorshifts for the pairs of ellipses, the thickness of the boundary outlines of the ellipses, and the type of graphical rendering, respectively. The implementation of function rotateEllipsePair[ ] uses linear subdivisions for five separate intervals. All of them are implemented with the single functions `linear[{e_,f_},{k_,n_},s_:1]:= e + s k (f-e)/n` where {e, f} represents the interval, {k, n} step number k in an n-element subdivision of the interval, and s a scaling factor.

Conclusion

One of the large surprises resulting from my investigations into these images has been that they fooled my eye and brain, and hopefully yours too, into three dimensions. Why that is must be related to the way that the eye perceives light and shadow expressed through color contrasts and the mechanism by which the brain constructs an image from those cues. Even though none of the images are even remotely associated with visual experiences in daily life or in nature there is no question in my mind that I see inherently three-dimensional figures and possibly objects from an alternate reality. The other huge surprise to me has been that the images are beautiful, full of artistic design elements. Their mathematical construction is obvious in many ways, but some of the underlying concepts remain hidden at first sight. I have used POEs as the single two-dimensional design element in these images, but I can see three obvious directions in which to prospect for new images: choosing other classes of curves that form the backbone for placing POEs, relaxing the orthogonality condition to arbitrary and even changing angles between the pairs of ellipses, and thirdly using closed planar curves other than ellipses as basic visual elements. I hope to be able to create many more images based on the principles outlined in this paper.

References

[1] Stewart James, *Calculus*, 5th ed. (2003), Thomson Brooks/Cole (For the mathematics of parametric and polar functions)

[2] Wolfram *MathWorld*, http://mathworld.wolfram.com/topics/Curves.html (April 14, 2012)
(For a large catalog of images and *Mathematica* code for plane and polar curves)