Point Symmetry Patterns on a Regular Hexagonal Tessellation

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Abstract

An investigation of point symmetry patterns on the regular hexagonal tessellation is presented. This tessellation has three point symmetry groups. However, the restriction to the hexagonal tessellation causes some symmetry subgroups to be repeated in ways that are geometrically unique and others that are geometrically equivalent, resulting in a total of 14 geometrically distinct symmetry groups. Each symmetry group requires a particular set of motif symmetries to allow its construction. Examples of symmetric patterns are shown for several simple motif families.

Introduction

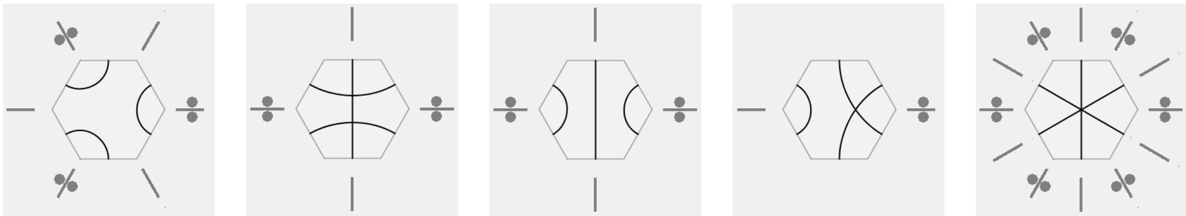
Throughout history, the symmetry found in regular polygons has been used to created interesting patterns. These symmetries have been mathematically analyzed and give rise to an important class of groups, namely the dihedral groups. The symmetry group of a regular polygon with n sides is given by the dihedral group of order $2n$, denoted D_n [2] and $*n$ using orbifold notation [1]. This group contains elements that correspond to rotations of the polygon and rotations following a reflection about a line through the center and one of the vertices. The cyclic groups, C_n, which contain only rotations, are important subgroups of dihedral groups.

Uniform tessellations, where the number and order of regular polygons meeting at a vertex remains constant throughout the tessellation, are a common decorative element for planar surfaces. The simplest uniform tessellations are the tessellations by squares, regular hexagons, and regular triangles. Each uniform tessellation of the plane by undecorated tiles has an underlying wallpaper or planar symmetry group. The tessellation of the plane using regular hexagons has the $*632$ wallpaper symmetry group using orbifold notation [1]. In this notation, the $*$ indicates the symmetry has points with dihedral symmetry. The numbers following $*$ indicate the order of the symmetry, with 6 indicating a dihedral group of order twelve (D_6), 3 indicating a dihedral group of order six (D_3), and 2 indicating a dihedral group of order four (D_2). The centers of each of these point symmetry groups are located at unique locations with respect to the hexagons as shown in Figure 2. The D_6 group fixes a point at the center of the hexagon, the D_3 group fixes a point at a vertex, and the D_2 group fixes a point located at the midpoint of the edge of a hexagon.

By decorating the tiles in a tessellation with simple motifs, one can create elaborate patterns in a modular manner. For example, the author has previously described a technique for creating interlace patterns by decorating the polygons in a regular tessellation using a simple motif using Bézier curves [3]. Each n-gon is decorated by simple cubic Bézier curve connecting pairs of edge midpoints. The set of the possible geometrically unique motifs (unique up to rotation) for decorating a hexagon with three arcs is shown in Figure 1. Decorating tiles with such motifs can also alter the symmetry group of a tessellation. This work presents the possible point symmetry groups possible for the regular hexagonal tessellation.

Methods and Results

Creating a point symmetric pattern from decorated regular hexagonal tiles depends on the symmetry groups of the hexagonal tessellation and the available tile decorations. The subgroup structure of the three dihedral



$D_3(D_{1_V})$ $D_2(D_{1_E}, D_{1_V})$ $D_2(D_{1_E}, D_{1_V})$ $D_1(D_{1_V})$ $D_6(D_{1_E}, D_{1_V})$

Figure 1: *The five possible geometrically unique motifs for decorating a hexagon with three arcs. Each hexagon is decorated with three Bézier curves that connect the midpoints of edges. The symmetry type of each motif is given and in parentheses are the bilateral (D_1) symmetry groups present for each motif. Here D_{1_V} indicates a mirror line passes through a vertex and D_{1_E} indicates a mirror line passes through an edge midpoint. Note that with this motif family, there is no motif having just D_{1_E} symmetry and there is no asymmetric (C_1) motif. The lines outside each hexagon indicate mirror lines and the dots represent a set of orbits for the indicated symmetry group.*

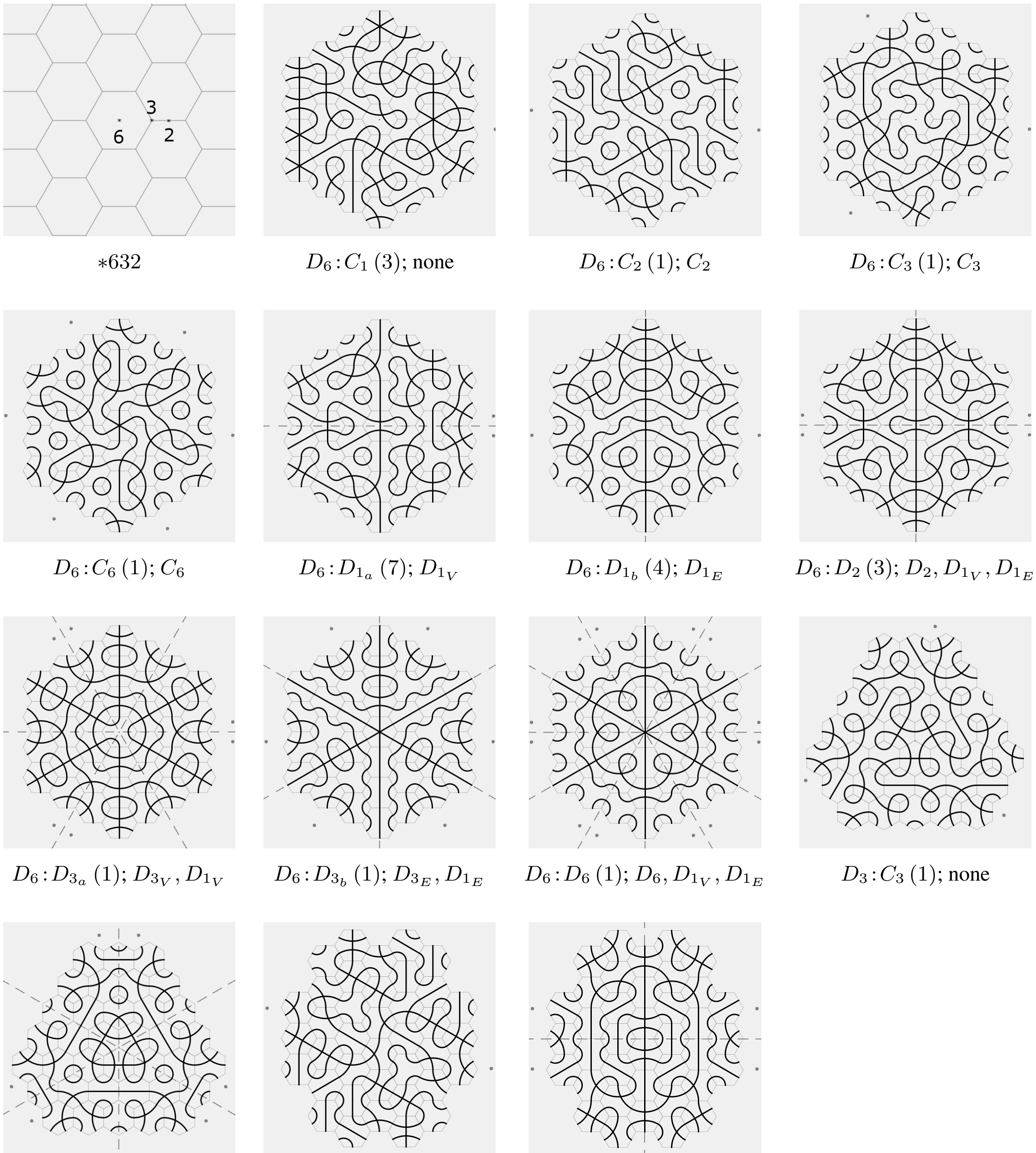
groups found in the hexagonal tessellation ($D_6, D_3,$ and D_2) give all possible symmetry groups. While multiple subgroups of the same type are present, this does not result in a repeated symmetry pattern in all cases. For example, the C_3 subgroup of D_6, denoted here as $D_6 : C_3$, fixes a point at the center of the hexagon while the C_3 subgroup of D_3, denoted $D_3 : C_3$, fixes a point at a vertex, resulting in two distinct symmetry patterns on the hexagonal tessellation. Likewise, the six D_1 subgroups of D_6 having bilateral symmetry fall into two geometric classes: the groups that reflect about a line through the midpoints of edges (D_{1_E}) and the groups that reflect about a line through opposite vertices (D_{1_V}). Of the 27 total subgroups (16 for D_6, 6 for D_3, and 5 for D_2), there are only 14 geometrically unique symmetry patterns for the hexagonal tessellation as shown in Figure 2.

Some point symmetry patterns require individual hexagonal tiles having specific tile patterns. For example, $D_6 : C_3$ requires a center tile with general C_3 symmetry, while $D_3 : C_3$ has no such requirement. Similarly, $D_6 : D_6$ requires a center tile with D_6 symmetry, field tiles with both D_{1_V} symmetry and D_{1_E} symmetry because there are mirror lines that bisect hexagons at edge midpoints and through opposite vertices. In the subgroups of the D_6 symmetry group, the motif of the center polygon must be selected to match the subgroup symmetry. In outer polygons where the center lies along a mirror line, the polygon motif must have D_1 symmetry: D_{1_V} symmetry if the mirror line also passes through a vertex and D_{1_E} if the mirror line passes through an edge midpoint. When selecting a motif, its symmetry type must be verified to match the requirements of a particular location. Each motif must also be rotated to match the subgroup symmetry of the tile location. For example, tiles surrounding the center tile in the $D_6 : D_6$ pattern require D_{1_E} symmetry and must be oriented correctly with respect to the mirror lines.

Figure 3 shows two examples with C_6 symmetry using related motif families. As with the previous examples, these patterns were created by drawing motif patterns on a symmetric hexagonal grid. The selected symmetry has a corresponding origin that is used to determine the set of equivalent hexagons from the standpoint of the symmetry group. The motif patterns were then selected for one tile and mapped via the symmetry operation to all other equivalent hexagons.

Discussion

The regular hexagonal tessellation has three parent point symmetry groups with a total of 27 subgroups. However, the restriction to the hexagonal tessellation causes some symmetry subgroups to be repeated,



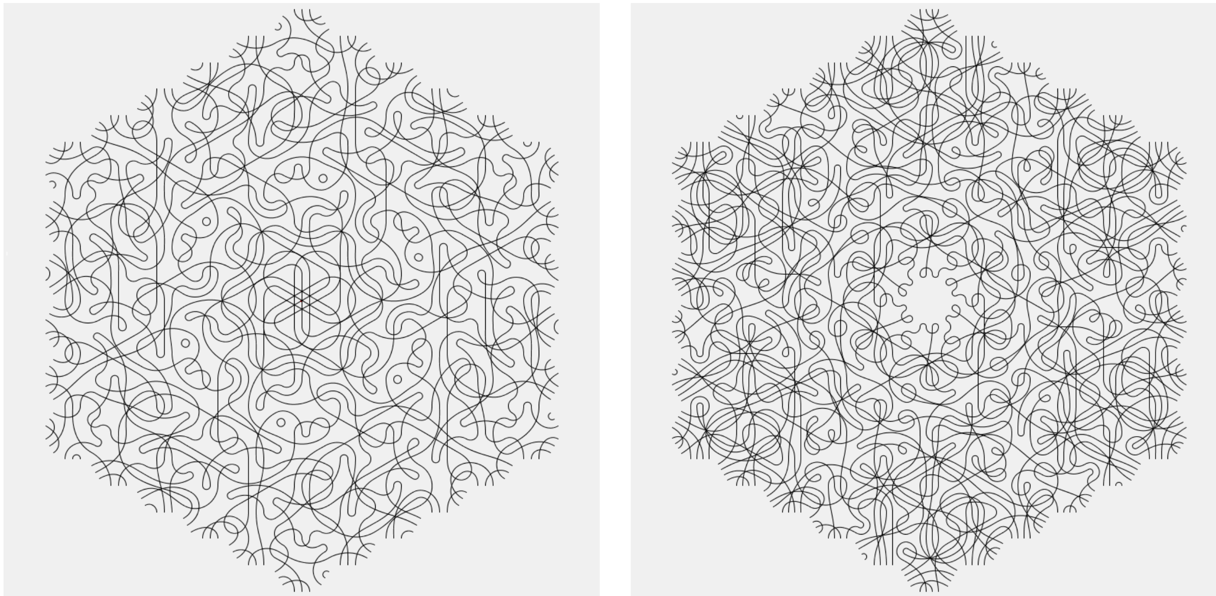
$*632$ | $D_6 : C_1$ (3); none | $D_6 : C_2$ (1); C_2 | $D_6 : C_3$ (1); C_3

$D_6 : C_6$ (1); C_6 | $D_6 : D_{1_a}$ (7); D_{1_V} | $D_6 : D_{1_b}$ (4); D_{1_E} | $D_6 : D_2$ (3); D_2, D_{1_V}, D_{1_E}

$D_6 : D_{3_a}$ (1); D_{3_V}, D_{1_V} | $D_6 : D_{3_b}$ (1); D_{3_E}, D_{1_E} | $D_6 : D_6$ (1); D_6, D_{1_V}, D_{1_E} | $D_3 : C_3$ (1); none

$D_3 : D_3$ (1); D_{1_V} | $D_2 : C_2$ (1); none | $D_2 : D_2$ (1); D_{1_V}, D_{1_E}

Figure 2 : *Example patterns using the geometrically unique symmetry groups on the tessellation by regular hexagons. In the $*632$ symmetry group, the center of each hexagon is the center point of the dihedral group D_6, each vertex is the center point of the dihedral group D_3, and each edge midpoint is the center point of the dihedral group D_2. Mirror lines are shown along with the orbit of an example point in the given symmetry group $G : H$, where G is the parent group (D_6, D_3, D_2) and H is the subgroup. The number of equivalent symmetries is shown in parentheses for each point symmetry group. The unique tile motifs (see Figure 1) required for each symmetry group are shown following the semicolon in each pattern name.*



3 arcs per edge 4 arcs per edge

Figure 3: *Example C_6 patterns on the tessellation by regular hexagons. In these figures, the motif's Bézier arcs connect endpoints evenly spaced along the edges of hexagons. The motif family for the figure on the left comprises arcs connecting three endpoints per hexagon edge (9 arcs per hexagon). The motif family for the figure on the right comprises arcs connecting four endpoints per hexagon edge (12 arcs per hexagon).*

resulting in only 14 geometrically unique point symmetric groups. Example point symmetric patterns for each of these 14 point symmetry groups were given. Note that there are two geometrically unique forms of D_1 symmetry. While these patterns look similar, $D_6 : D_{1_a}$ requires tiles that have D_{1_V} symmetry and $D_6 : D_{1_b}$ requires tiles that have D_{1_E} symmetry. Likewise, there are three geometrically unique forms of D_3 symmetry and two geometrically unique forms of D_2 symmetry. Each of the point symmetry groups requires a particular set of motif symmetries to allow its construction. Any motif pattern family having tiles with appropriate symmetries can be used to create symmetry patterns. The bounded nature of symmetric patterns created in this manner can have more visual appeal than conventional planar tessellations.

Acknowledgments

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