Polyhedra on an equilateral hyperboloid

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Abstract

The paper gives examples of polyhedra inscribed in a (equilateral) hyperboloid, because this surface can be seen as negative curvature counterpart of a sphere. There is a historical argument: Kepler used polyhedra inscribed in spheres as a model for the orbits of planets, and so, since many comets follow a hyperbolic path, their orbits might be compared to polyhedra in hyperboloids. Thus, square, penta-, hexa- and heptagrammic crossed prisms are proposed, as well as an (extended) 'hyperbolic cuboctahedron' with a recognizable hourglass aspect. The approach is not exhaustive, but it might nevertheless inspire poetic mathematicians to develop a 'Comet Mysterium', or artists to make creations using the hyperboloid.

Kepler's Mysterium for Comets

Around 1600 Johannes Kepler described the distance relationships between the planets in terms of the Platonic solids. Approximately, planets move in orbits enclosed in spheres and as the latter also circumscribe the classical polyhedra, it seemed an ingenious concept to link the five Platonic solids to the six planets known at that time. Today, we know there are more than six planets, but Kepler's 'Mysterium Cosmographicum' remains an inspiring model: many comets follow a hyperbolic path, and thus the present paper proposes models with vertices on a hyperboloid. Some poetic mathematicians might feel inspired to develop a 'Comet Mysterium' (see fig. 1).
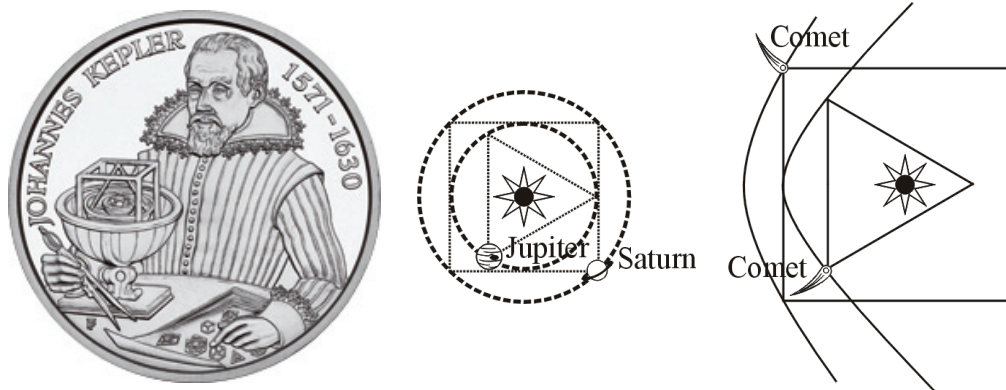

Figure 1: *Kepler's model on an Austrian coin (left); if Jupiter and Saturn would fit in Kepler's spherical model (middle), some comets might fit in a similar hyperboloid model (right).*

Counterpart of the Sphere

The hyperboloid of one sheet generated by revolution of an equilateral hyperbola around a suitable axis will be considered as an analogue of the sphere. There are many reasons to do so. For instance, in 2D-geometry the hyperbola is in natural correspondence to the ellipse. Both conic sections arise in practice in many ways; ellipses describe the orbits of planets and comets, but hyperbolas do so for some other comets. A hyperbola has two open branches becoming straight further out from the center of the hyperbola where they tend to the asymptotes. The latter intersect at the hyperbola's center and if this happens at a right angle, the hyperbola is said to be 'equilateral' (or 'rectangular' or 'orthogonal'). The equilateral hyperbola with equation $x^2 - y^2 = 1$ is the set of all points $P(\cosh(u), \sinh(u))$, just as the circle $x^2 + y^2 = 1$ describes all points $P(\cos(\theta), \sin(\theta))$ (see fig. 2a).
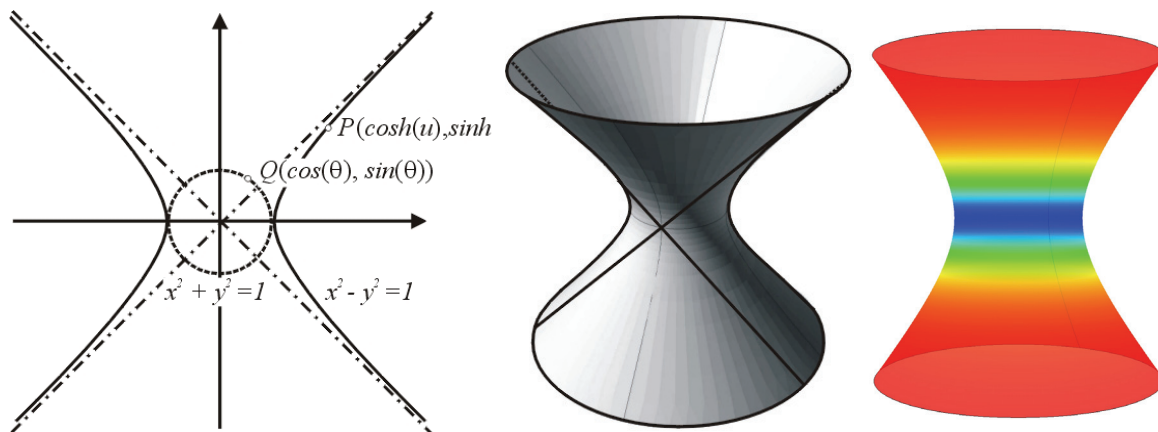

Figure 2: *The equilateral hyperbola (left); the hyperboloid of revolution with 2 straight lines on the surface showing it is a double ruled surface (middle); the change in curvature on this surface (right).*

Revolving the equilateral hyperbola with equation $x^2 - y^2 = 1$ around the y-axis creates a hyperboloid of revolution described by the equation $x^2 + z^2 - y^2 = 1$. Cooling towers of nuclear power plants are often quoted as typical examples, though they do not obey the hyperboloid shape exactly, due to modifications to reduce wind resistance (see [3]). The hyperboloid of one sheet is a ruled surface: it can also be obtained by revolving a skew line, a generator, about the axis, implying it can easily be built with straight beams. Moreover, it is a doubly ruled surface (see fig. 2b), as the revolution of another straight line can create the same surface. There are also reasons for not considering the hyperboloid of one sheet as a counterpart of the sphere: its (Gaussian) curvature is negative and becomes almost 0 'far' from the origin (see fig. 2c), whereas the sphere's curvature is the positive constant +1. Thus, E. Beltrami's 'pseudosphere' of radius 1 and constant curvature −1, could be a better negative counterpart of the (unit) sphere. Nevertheless, the present paper will consider the equilateral hyperboloid of one sheet as the equivalent of the sphere.

Polyhedra on a Non-Equilateral Hyperboloid

It has been noticed some polyhedra have their vertices on a hyperboloid. Robert Dawson (see [2]) mentions the cases of the stellar octangula and the pentagrammatic crossed antiprism. The latter is formed by two parallel pentagrams rotated 36° and linked by equilateral triangles; its vertices lay on a sphere (see fig. 3a). In addition, the solid has its vertices on a hyperboloid since a skew edge can be used as generator. Its revolution around the axis through the centers of the parallel pentagrams yields a hyperboloid (see fig. 3b). However, the thus created hyperboloid is more slender than an equilateral hyperboloid (see fig. 3c and 3d).
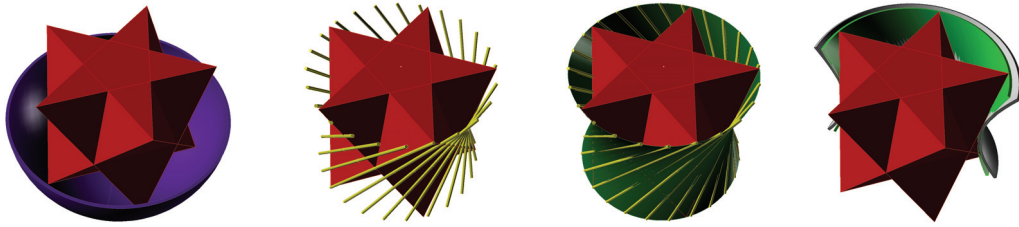
Figure 3: *A pentagrammatic crossed antiprism can be inscribed in a sphere (left). Its skew edges generate a hyperboloid (middle), but when compared to an equilateral hyperboloid the difference is significant (right).*

Dawson's example can also be applied for a triangular antiprism, that is, a prism formed by two parallel equilateral triangles, rotated 60° and linked by equilateral triangles, that is, an open octahedron. Its vertices lay on a sphere as it is one of the Platonic solids. The hyperboloid created by revolution of one of the edges again is not equilateral (see fig. 4).
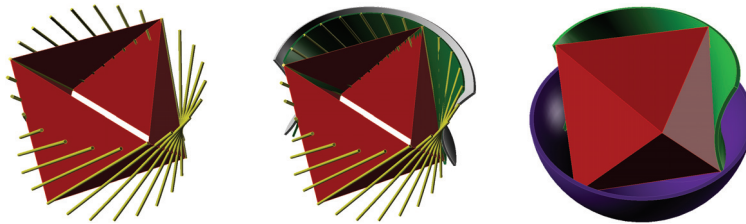

Figure 4: *When edges of an 'open' octahedron revolve (left), they create a hyperboloid, which is more slender than an equilateral hyperboloid (middle); thus, an octahedron fits in a sphere as well as in a hyperboloid (right).*

Combining two of those inscribed octahedrons yields a 'hexa-grammic crossed antiprism' similar to Dawson's pentagrammatic antiprism (see fig. 5a). In a similar way, a hepta-grammic crossed prism can be inscribed in a (non-equilateral) hyperboloid (see fig. 5b). The procedure can be extended to nona-grammic crossed prisms and so on. Here we also considered the case of an open antiprism with a square basis (see fig. 5c), and its related octa-grammic crossed prism, but it does not yield an equilateral hyperboloid either (see fig. 5d).
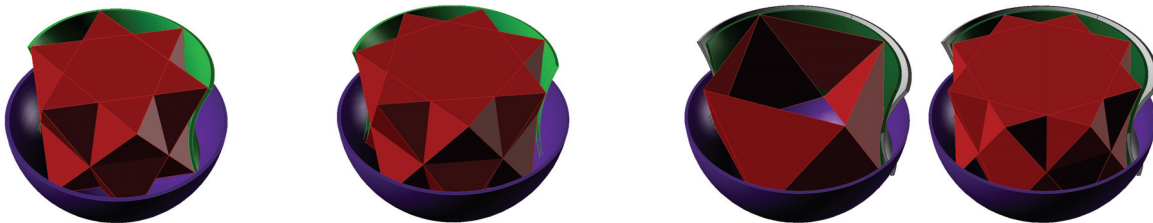

Figure 5: *A hexa-grammic prism (left) and a hepta-grammic prism in a (non-equilateral) hyperboloid (middle). A square antiprism lays on a hyperboloid which is not equilateral, and combining them gives an octa-grammic crossed prism (left).*

Another application of the procedure is to extend the generators. For instance, an open octahedron can create a polyhedron with 6 squares and 12 equilateral triangles (see fig. 6a). It is a good approximation for a (non-equilateral) hyperboloid (see fig. 6b), because all vertices lay on the hyperboloid. It can be closed by two regular hexagons and has an hourglass shape (see fig. 6c).
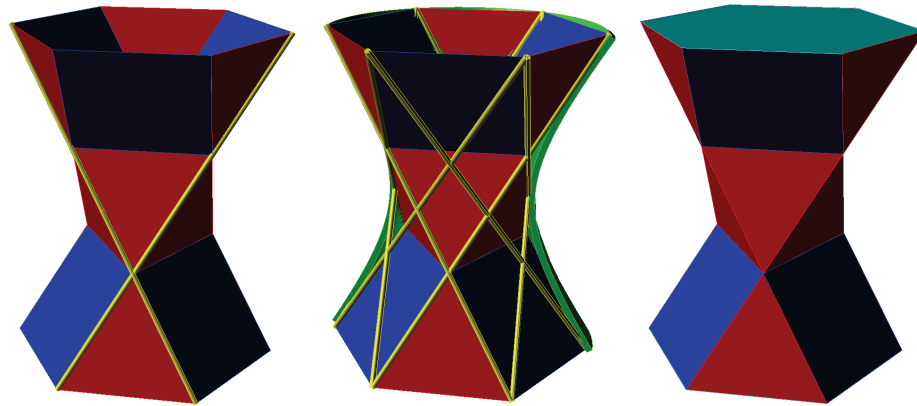
Figure 6: *The open octahedron extended along the generators (left), which revolve to a hyperboloid (middle); thus, when covered by hexagons, the closed polyhedron has all its vertices on a hyperboloid (right).*

Polyhedra on an Equilateral Hyperboloid

The hyperboloids in the previous paragraph were not equilateral. Ellipsoids can be considered as their counterparts in positive curvature. If we want to consider a counterpart of a sphere, and polyhedra inscribed in it, we may want to restrict ourselves to equilateral hyperboloids. As mentioned above, Dawson (see [2]) also gave the case of the stella octangula, a compound of two tetrahedra. Its vertices all lay on a sphere (see fig. 7a), but since some edges are orthogonal, it also fits in a cube (see fig. 7b). Thus, with those edges as generators, the hyperboloid created is equilateral (see fig. 7c). From the stella octangula example it follows that a tetrahedron fits in an equilateral hyperboloid. It follows that another polyhedron can be derived from it, laying on an equilateral hyperbola. It can be covered it by 2 squares (see fig. 8). The polyhedron could be called a 'hyperbolic cuboctahedron', because a rearrangement of the upper and lower parts yields an open 'spherical' cuboctahedron.
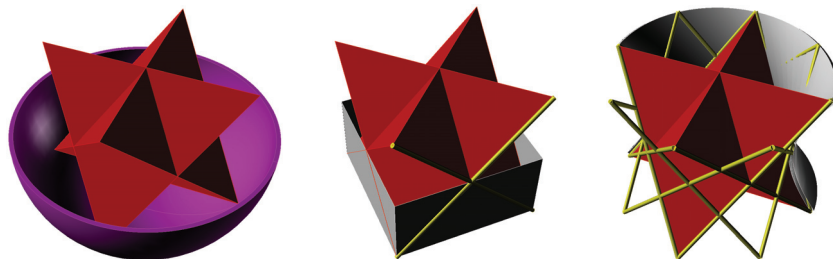

Figure 7: *The stella octangula in a sphere, in a cube, and in an equilateral hyperboloid.*
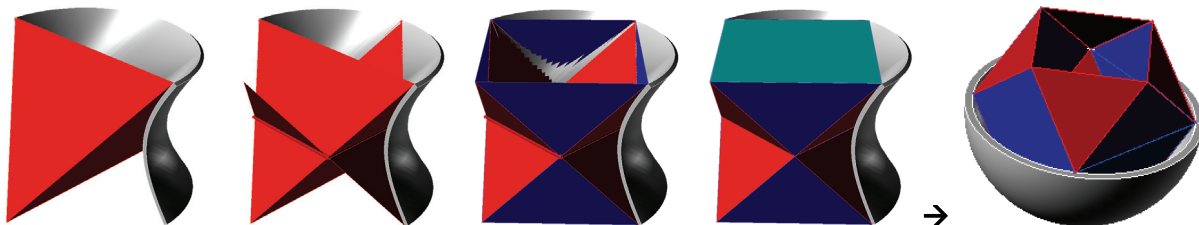

Figure 8: *Derivation of another polyhedron on an equilateral hyperboloid, which can be transformed into an open cuboctahedron.*

'In' or 'on' a Hyperboloid

The 'hyperbolic cuboctahedron' can be extended by squares and equilateral triangles (of a different size) so that a better lookalike of an equilateral hyperboloid is obtained (see fig. 9a). When the middle part, that is, the 'hyperbolic cuboctahedron', is replaced by a cube, another hourglass type polyhedron is obtained (see fig. 9b). This polyhedron may be called a 'hyperbolic rhombicuboctahedron' since a rearrangement of its faces transforms it into a 'regular spherical' rhombicuboctahedron (again, when disregarding some of the faces). This rhombicuboctahedron played an important role in the history of polyhedra (see [4].)
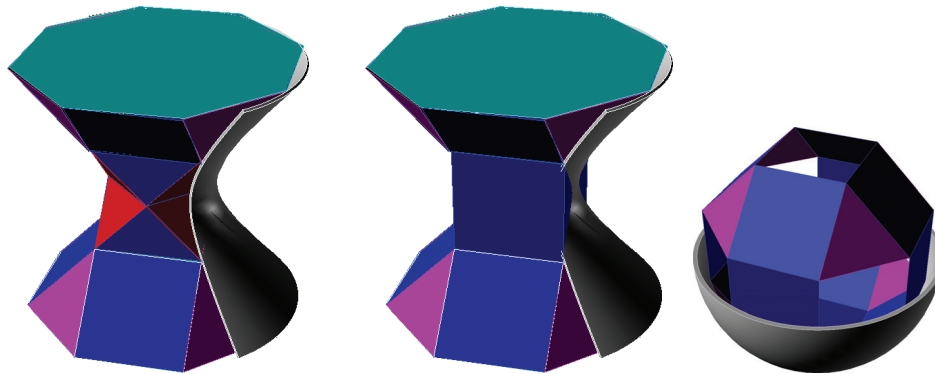
Figure 9: *The extended 'hyperbolic cuboctahedron' (left), the 'hyperbolic rhombicuboctahedron' (middle) and the explanation for its name (right).*

However, the 'hyperbolic rhombicuboctahedron' does not entirely lie inside the (equilateral) hyperboloid, since the middle cube crosses the surface. Still, it seems well acceptable that a polygon with vertices on a negatively curved surface crosses the surface, even if all vertices are on the surface. In fact, this is typical for a negatively curved surface. Indeed, surfaces with a positive (Gaussian) curvature have a bowl-like shape: any transversal section looks like the letter 'n' when turned around in some direction. It contrasts with surfaces like planes, cylinders and cones that have zero curvature: they are flat in at least one direction. On the other hand, saddle surfaces for instance have a negative curvature: a transversal section looks like the letter 'n' in some directions, and like the letter 'u' in other directions (see fig. 10). Thus, for surfaces with a negative curvature, there is no good reason to restrain the polyhedra on a hyperboloid to the restriction that they be entirely situated 'inside' the hyperboloid.
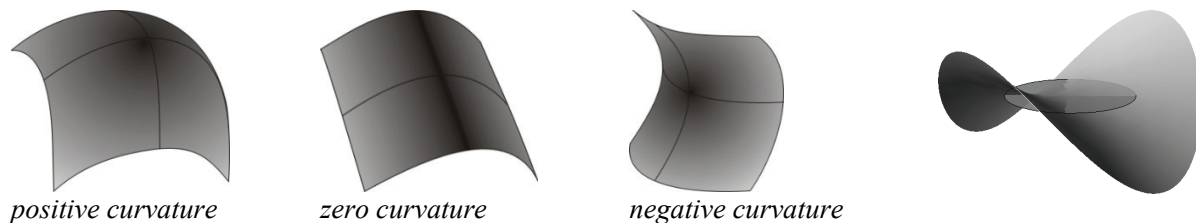
positive curvature   *zero curvature*   *negative curvature*
Figure 10: *Surfaces with different curvatures (left, middle), and a disk with center in a point with negative curvature (right).*

Summarizing, several cases can occur: the tetrahedron, for instance, fits *on* and *in equilateral* hyperboloid (see fig. 11a), while the octahedron lies *on* and *in* a *non-equilateral* hyperboloid (see fig. 11b), while a cube is *on* yet not *in* an *equilateral* hyperboloid (see fig. 11c).
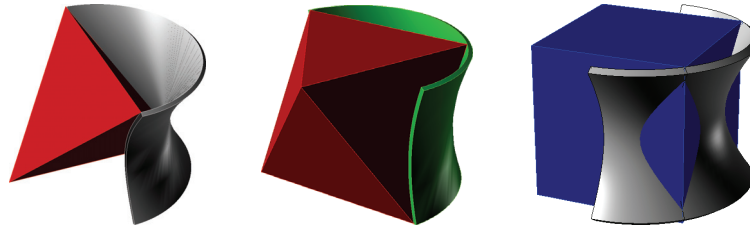
Figure 11: *The tetrahedron, the octahedron, and the cube, compared to an (equilateral) hyperboloid.*

Once the natural condition remains that only the vertices lie on the hyperboloid, more possibilities emerge. For instance, even two tetrahedrons with one common edge fit on an hyperboloid, and they can be covered by four rectangular triangles. Or else, a prism with an equilateral triangular base, a translated copy and three square faces fits as well. Intuitively, the previous examples don't look pleasant because they don't mimic the hour glass shape of the hyperboloid. Yet, starting with an equilateral triangle on the middle brim of the equilateral hyperboloid, two polyhedra ending in hexagons can be erected. They fit exactly on the equilateral hyperboloid, and, moreover, after some rearrangement they transform into a cuboctahedron (see fig. 12).Thus, the polyhedron could be called a 'hyperbolic cuboctahedron'.
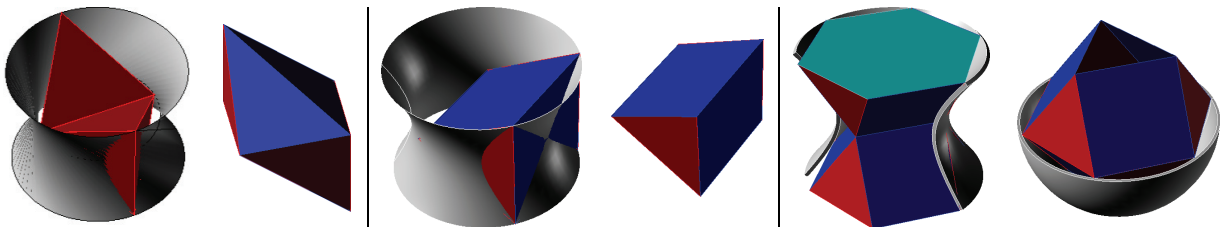

Figure 12: *A 'hyperbolic Johnson solid' (left), a triangular prism with square faces (middle) and a 'hyperbolic cuboctahedron' compared to a spherical cuboctahedron (right).*

Open Polyhedra on a Hyperboloid

In figures 9 and 12 only the vertices of the polyhedra were on the hyperboloid, but the solids had a more pleasing hour glass shape. They were 'covered' by respectively regular octahedrons and hexagons, because tradition suggests that polyhedra to be closed shapes, that is, 'solids', but in the context of hyperboloids there is no reason for this either, since a hyperboloid is not a 'closed' shape itself – in contrast to a sphere or an ellipsoid. Thus, the following disposition seems interesting: eighteen equilateral triangles grouped on an equilateral hyperboloid (see fig. 13).
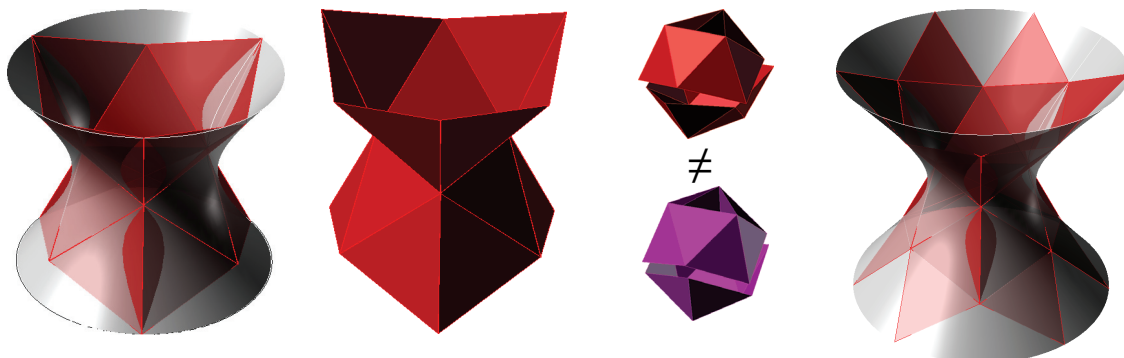

Figure 13: *Eighteen (left, middle) equilateral triangles on an equilateral hyperboloid, but they do not reshape into an icosahedron (middle); thirty triangles form an even more open shape (right).*

This model nicely corresponds to Thurston's ideas about models for the hyperbolic plane (see [1], [7]). The model is not simply connected, but still, eight triangles meet in each of the three vertices of the middle ring. We could, as in figures 9 and 12, regroup the upper and lower part in order to find a similarity with a known polyhedron. However, the two parts do not fit together now, though they seem to form an open icosahedron (see fig. 13). The model can be extended to thirty triangles on an equilateral hyperboloid but these additional triangles have only one edge in common. Of course, the procedure can be continued, but the additional triangles will again have only one edge in common.

One can wonder if it is possible to let *seven* equilateral triangles meet in a vertex for a set of triangles with vertices on a hyperboloid, because this would also fit nicely in Thurston's model. The open octahedron of figure 4 can be extended by a ring of equilateral triangles to form a surface that approximates a hyperboloid, instead of with squares and triangles, as in figure 9. However, now the top vertices do not meet on the hyperboloid. After all, this is not so surprising, because the curvature on hyperboloid is not constant (see fig. 14).
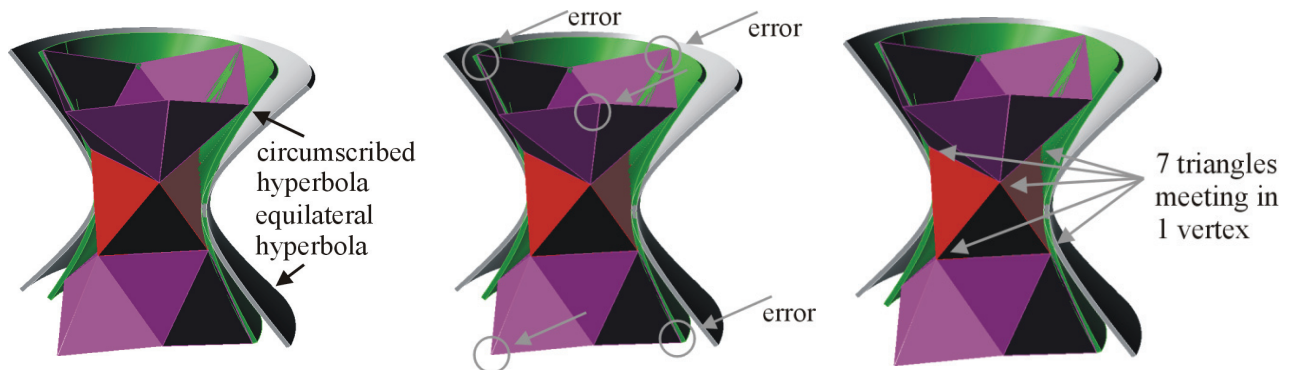

Figure 14: *An octahedron with equilateral triangles approximates a hyperbola (left), but the extreme vertices do not meet on the hyperboloid (middle); it would have been an example of 7 triangles meeting in a point to approximate a hyperboloid (right).*

Artistic Explorations with Hyperbolic Polyhedra

Finding polyhedra on a hyperboloid can open up new amusing results in the thousands-years-old field of polyhedra. Consider the construction of the well-known Fuller domes: when an icosahedron is at the start of the triangulation of the circumscribed sphere, there will always be twelve groups of triangles meeting by five, corresponding to the twelve vertices of the icosahedron, apart from the many (not necessarily equilateral) triangles meeting by six. The case of the soccer ball with its twelve pentagons and twenty hexagons is a well-known related result (see [5]). This Fuller construction can be applied to the extended octahedron of figure 14, and since there were six vertices where seven triangles met, there will be six groups of triangles meeting by seven (see fig. 15). For sure, this is an interesting application for architects (see [6]).
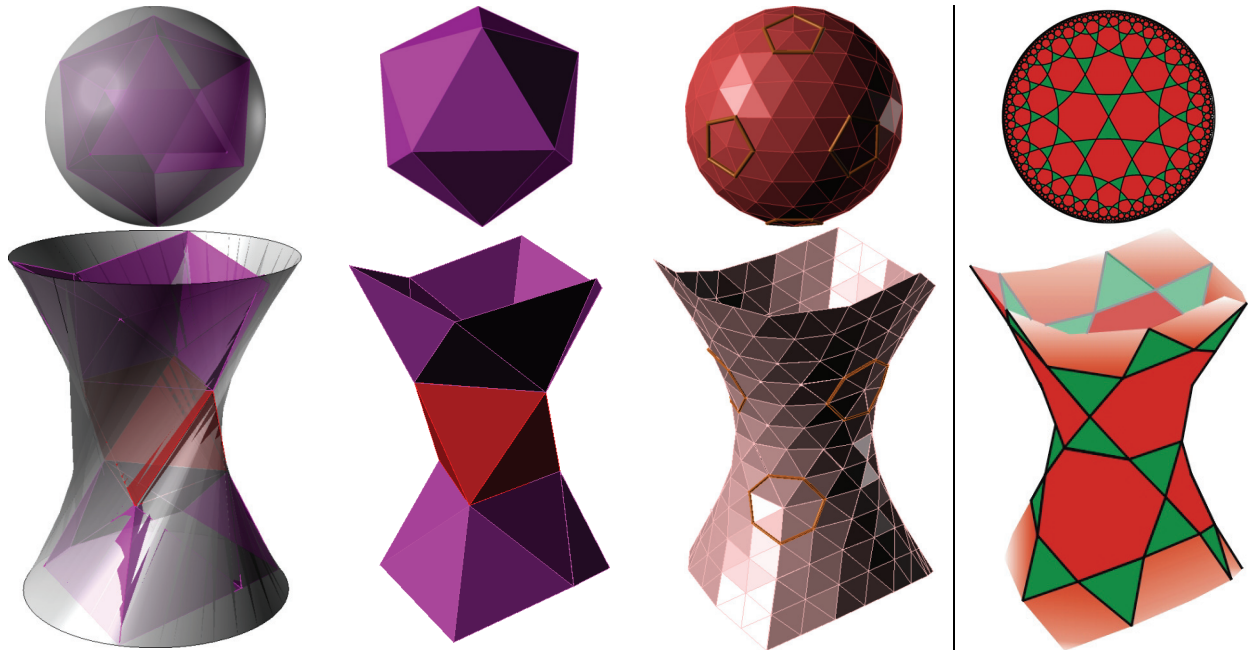
Figure 15: *Above: on a spherical Fuller dome, constructed from an icosahedron, one distinguishes twelve groups of triangles meeting by five; a Poincaré disk model for the hyperbolic plane can represent the (3,3,7,7) case (drawing by R. Sazdanovic, University of Pennsylvania). Below: on a 'hyperbolic' Fuller dome, six groups of triangles meet by seven; the (3,3,7,7) case on a hyperboloid.*

Another use of the models given here is an alternative representation of tessellations, such as for the (3, 3, 7, 7) case given in figure 15. Just as spherical tessellations are represented on a sphere, and Euclidean tessellations on the plane, hyperbolic tessellations can now be shown on a hyperboloid.

References

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