Polyhedral Modularity in a Special Class of Decagram Based Interlocking Star Polygons

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Abstract

The main effort in this article is to study a series of Persian mosaic designs that have been illustrated in scrolls or which decorate the surfaces of old structures. The common element in these designs is a special ten pointed star polygon. This special concave polygon, called a decagram for convenience, is the dominant geometric shape of a series of polyhedral tessellations that all consist of the same common motifs.

1. Introduction

From a few documents left from the past, it is evident that the designers of patterns on the surfaces of medieval structures in Persia and surrounding regions, were well-equipped with a significant level of knowledge of applied geometry. Nevertheless, none of them exhibits the same level of effort or interest in providing the pure side of the subject by proving theorems and establishing mathematical facts about such designs. The main concern of a designer or a craftsman was to present a visual harmony and balance, not only in deep details, but also as a whole. But the steps taken for creating such designs, which can be discovered today give little room to a researcher to assume accident or pure experience. Some techniques are only acquired and understood by mathematicians. An individual with the knowledge of such detailed techniques is a mathematician, artist or not.

2. Tiling with a Special Ten Pointed Star Polygon (*decagram*)

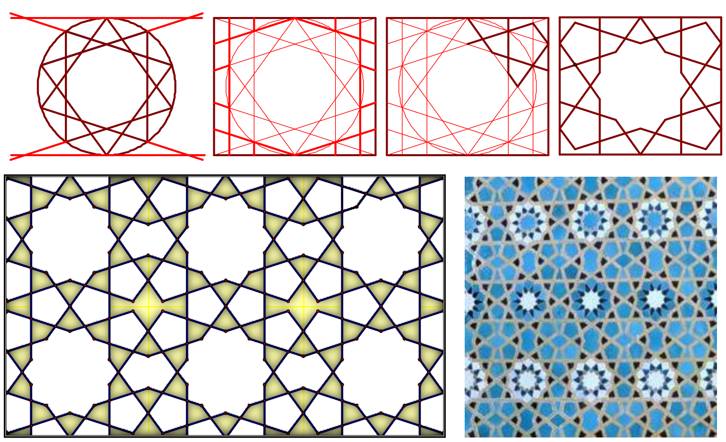


Figure 1: (U) *The compass-straightedge construction of a decagram tile* (LL) *The tessellation of the tile,* (LR) *The tessellation of the pattern on an existing wall in Masjid-i-Jami, Kerman in Iran.*

Figure 1 exhibits a method for creating a tiling design [4]. In this method, the starting point is a (10, 3) Star Polygon – a figure created from connecting every third vertex in a set of 10 equally spaced points on a circle in one direction in one stroke (the star inside the circle on the upper left corner of Figure 1). By extending some of the segments that constituted the (10, 3) Star Polygon and intersecting them with the lines perpendicular to some other segments, one achieves the construction of the rectangular frame and other necessary line segments inside the frame as is on the upper right corner of Figure 1. This rectangular tile tessellates the plane and creates a pleasing series of stars that are ordered in columns and in rows.

In the tessellation presented in Figure 1, the special ten pointed star polygon, which is the dominant geometric shape of the tessellation, can be created independently through the rotation of two concentric congruent regular pentagons with a radial distance of 36° from each others' central angles (Figure 2(a, b)). Such a rotation with a radial distance of 45° for two squares creates the attractive and overwhelmingly used 8 pointed star polygon (call it *octagram*) that has appeared in many geometric designs and tilings as cross-octagram tessellations in Persian architecture and around the world (Figure 2(c, d, e)). So we may assume that creating a tile design using this 10-pointed star polygon (call it a *decagram* for convenience) was a challenge for designers, compared to a not-so-complicated and straight forward cross-octagram tessellation that can be defined in a square-shaped frame.

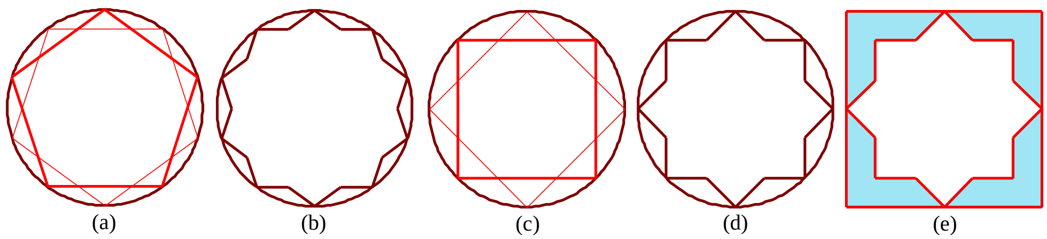
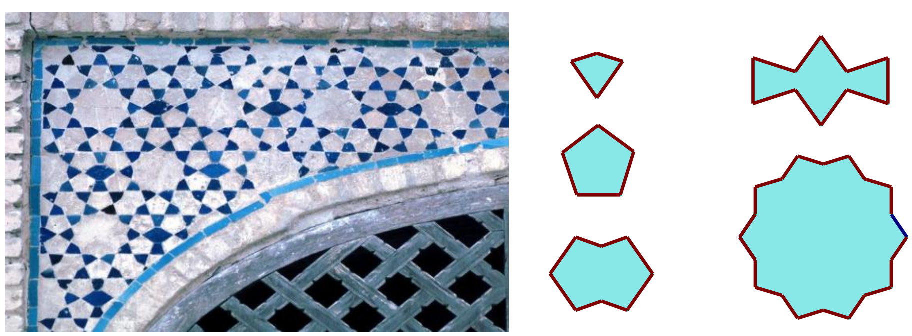


Figure 2: (a) *Two concentric pentagons*, (b) *generated decagram*, (c) *two concentric squares*, (d) *generated octagram*, (e) *a tile for the cross-octagram tessellation.*

Studying documents from the past reveals that in most cases the fundamental regions, similar to what is shown in Figure 1.UR, were shapes stored in scrolls (*tumār*) and booklets (*daftar*) as designs used in the executions of the actual tiling on the surfaces or as geometric experimentations of interlocking star-polygon patterns. Such a fundamental region was called a knot (*girih*) in Persian architecture.

Figure 3: (L) *Imamzeda Darbi Islam, Isfahan in Iran*, (R) *The five Sâzeh module tiles.*



The existing tessellation on the wall of a Persian structure in Figure 3.L includes a decagram motif. There are other shape tiles that constitute the tiling. In fact there are exactly five motifs (modules). Figure 3.R presents these modules. They are called *Muarraq* (معرق), an Arabic word, in Iran. The Arabian-Andalusian word for these hand-cut pieces of glazed ceramic tiles is *Zellij*. In this script they are called *Sâzeh* module tiles (سازه, structure in Persian).

These modules have their own specific Persian names: *Torange* (the quadrilateral tile), *Pange* (the pentagonal tile), *Tabl* (the concave octagonal tile), *Sormeh Dân* (the bow tie tile), and *Shamseh* (decagram tile).

Comparing the two tessellations in Figures 1 and 3, one may notice that, despite the fact that the individual Sâzeh modules used in both tessellations are identical, they are very different tessellations.

The following solution for the geometric construction of the tiling in Figure 3, with some revisions in the process, comes from the professional artisan, Maheronnaqsh, who inherited his profession from his ancestors of several centuries, who had the most access to original artisans' repertories of the past [12]:

Divide the right angle $\angle A$ into five congruent angles by creating four rays that emanate from *A*. Choose an arbitrary point *C* on the second ray, counter-clockwise, and drop perpendiculars from *C* to the sides of angle $\angle A$. This results in the rectangle *ABCD* along with four segments inside this rectangle, each with one endpoint at *A* and whose other endpoints are the intersections of the four rays with the two sides of *CB* and *CD* of rectangle *ABCD*. Find *E*, the midpoint of the fourth segment created from the fourth ray. Construct an arc with center *A* and radius *AE* to meet *AB* on *F* and the second ray on *G* (the second segment is now part of the diagonal of the rectangle). Make a line, parallel to *AD*, passing through *G*, that intersects the first ray at *H* and the third ray at *I*. Line *FH* passes through point *E* and meets *AD* at *J*. Construct a line, passing through *J*, that parallels the third ray. Also construct line *EI*. From *F* make a parallel line to the third ray to meet the first ray at *K*. Construct segments *GK*, *GL*, and *EM*. Find *N* such a way that *GI* = *IN*. Make a line through *N*, parallel to *GK*, to intersect the line emanate from *J*, to find *P* to complete the regular pentagon *EINPJ*. *Line DN* meets the perpendicular bisector of *AB* at *Q*. From *Q* construct a line parallel to *FK* to intersect ray *MI* at *R* and then complete the figure (Figure 4.L).

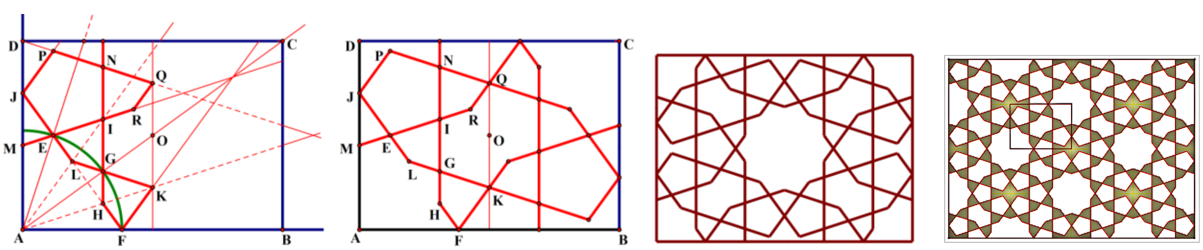


Figure 4

Using *O*, the center of the rectangle *ABCD*, as a center of rotation for 180°, one can make the fundamental region for the tiling in Figure 3. The last two images in Figure 4 show a tile and its tessellation, which is the tiling design in Figure 3.

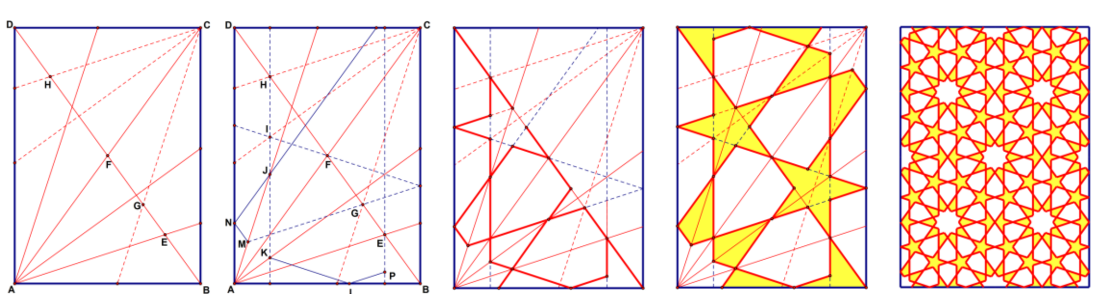


Figure 5

The construction method in Figure 4, which uses a radial grid approach, as a method used in the medieval time, is supported by some images along with their construction instructions, recorded in the *Interlocks of Similar or Complementary Figures*, a seven hundred year old Persian mathematics document [1]. Figure 5 is an example for a step-by-step construction of a star pattern design using radial grid approach created by the author.

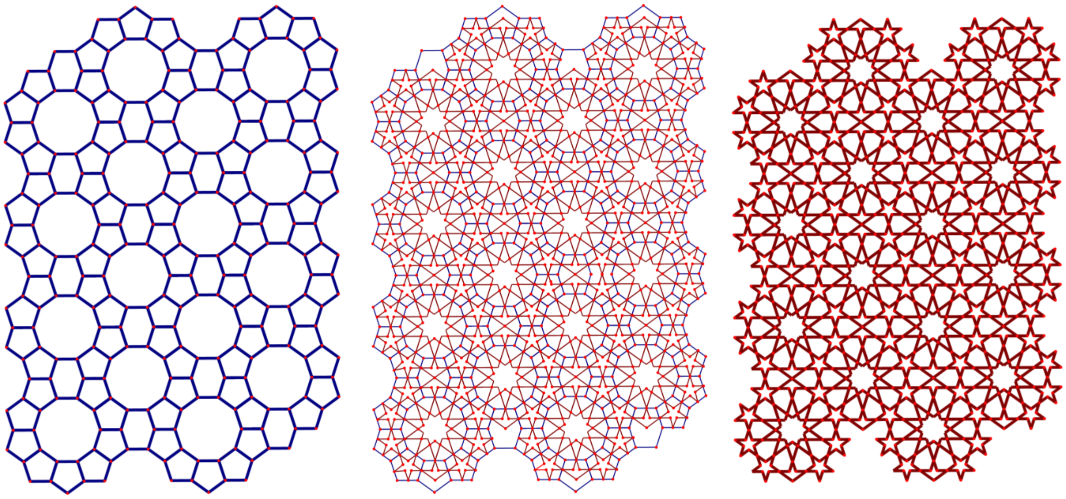


Figure 6

In some literature, another technique, "polygons in contact" (PIC), has been introduced [9], which is explained in some recent articles [2, 3]. This is another system for which there is evidence of historical use by designers [2, 3]. Figure 6 from left to right exhibits this technique starting from the underlying polygonal network ending at the final pattern, which is the same pattern as is constructed in Figure 5.

3. A Tessellation from Mirza Akbar Architectural Scrolls and Its Construction

The two different tessellations in Figure 1 and 3 that have been made from the same set of Sâzeh modules raise the question: Are there more tessellations that are made from the same set of decagram and its interlocking polygons?

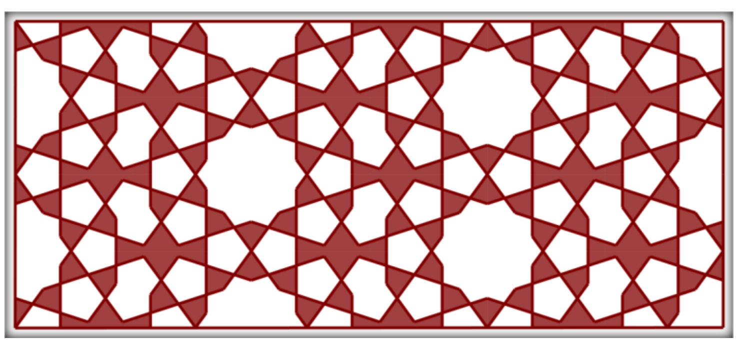


Figure 7: *A tessellation from the Mirza Akbar collection.*

The image in Figure 7 is an exact rendering of a design illustrated in the Mirza Akbar collection. In this tessellation, the decagrams are farther apart from each other. Using the steps involved in the construction of the design in section 2, we can find a traditional radial solution for this tessellation.

It is not difficult to discover that the fundamental rectangle for this tilling has a longer length compared to the rectangle in Figure 4. So starting with the radials that divide the right angle into five congruent angles the arbitrary point *P* was selected on the first ray counter-clockwise (rather than the second ray in the previous problem). For the radius of the circle inscribed in the decagram, one half of the segment created from the third ray, segment *AM*, was selected. Then a similar approach was taken to create the tiling in Figure 7. Figure 8 illustrates a step-by-step visual solution to the problem.

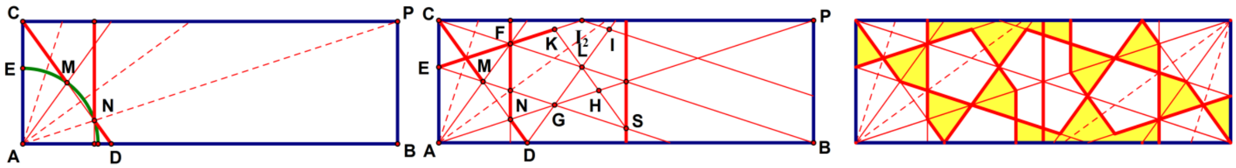


Figure 8

4. A Square Girih for Constructing a Tessellation

Locating an arbitrary point on any of the rays that divide the right angle into five congruent angles and dropping perpendiculars to the sides of the right angle results in only two different rectangles:

I. Selecting the arbitrary point *C* on the first ray and dropping the two perpendiculars *BC* and *CD* to the sides of right angle $\angle A$ results in the rectangle *ABCD* (Figure 12.L), where the relationship between its diagonal *AC* and side *BC* is $AC/BC = 2\phi = 1+\sqrt{5}$, where ϕ is the Golden Ratio. Therefore, $AB/BC = \sqrt{5+2\sqrt{5}}$.

II. Selecting the arbitrary point *F* on the second ray will result in the rectangle *AEFG* (Figure 12.L), which is the Golden Rectangle: $AE/EF = \phi$.

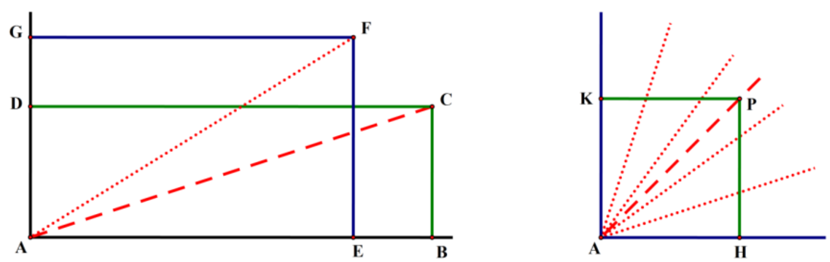


Figure 9

Now the question is whether using the same technique as mentioned above, are we able to come up with a new pattern composed from all five Sâzeh modules in Figure 3.R?

An interesting problem would be to consider the square as the girih solution. Obviously, none of the rays that divide $\angle A$ into five congruent angles helps directly. Choosing *P* as an arbitrary point on the angle bisector of $\angle A$ and constructing square *AHPK* cannot help us either (Figure 9.R) . Nevertheless, we are able to obtain a solution, if we start with an arc, with center *A*, and an arbitrary radius. This arc cuts the rays in certain points that are used to find a solution. The following images in Figure 10, starting from the upper left and ending at the lower right, demonstrate a step-by-step solution to this problem by the author.

Figure 11 is a tessellation that is created from the five Sâzeh modules based on the square girih in Figure 10. This artwork was exhibited in the Joint AMS-MAA Mathematics Meeting in Boston, Massachusetts, USA, in January 2012 [5] and in the 2012 Bridges Mathematical Art Exhibition at Towson University, Maryland.

Adding the tiling in Figure 11 to the above three tilings, illustrated in figures 1, 4, and 7, makes a set of four different mosaic patterns, each made from the aforementioned five Sâzeh modules. A curious reader may want to know whether more tessellations can be formed from this set of modules. The same curiosity may have promoted the craftsmen-mathematician of the past to look for new solutions that are not necessarily, at least in part, based on the compass-straightedge constructions.

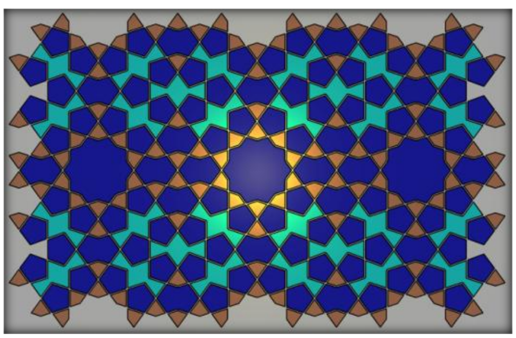


Figure 10

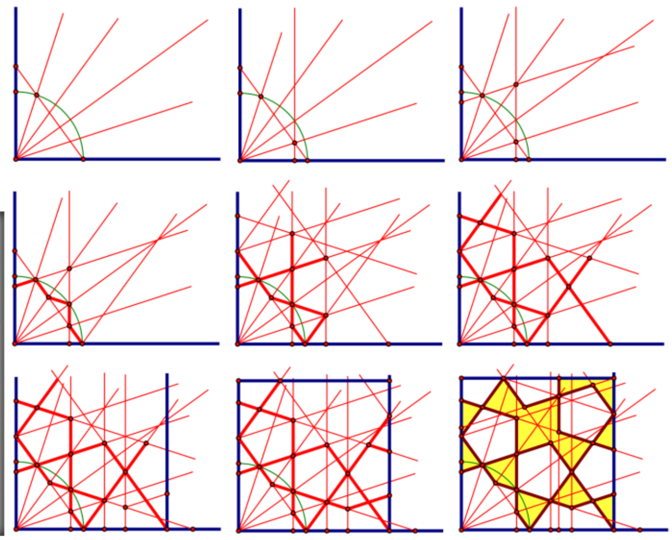


Figure 11

5. Modularity Approach in Mosaic Designs

5.1. Modularity based on color contrast: Figure 12 *Kharragan I* (January 2011) is an artwork by the author based on a design on one of the 11th century twin tomb towers in Kharraqan, western Iran. The artwork demonstrates two different approaches that are assumed to have been utilized centuries ago to create the layout of the pattern, which is at the center of the artwork. From left to right, the artwork exhibits the construction of the design based on a compass and straightedge. From right to left, we see another approach, the Modularity method based on color contrast, to construct the same design using cutting and pasting of tiles in two colors. These two methods of constructions were presented at [16]

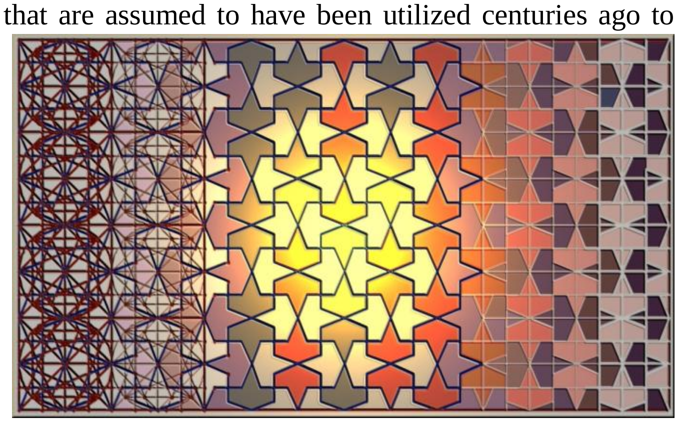


Figure 12

5.2. Modularity based on motifs formed from the combination of polygons. Figure 13.M, *Hope* (December 2008) is an artwork by the author [7], which is based on the modularity concept using two triangles that each have been composed from smaller triangles and rhombuses in three colors. The actual tiling adorns a wall of *Bibi Zinab* Mausoleum in Isfahan, Iran. Notice that in Figure 13.L, except for the corners, the two compound triangles (girih modules) are in opposite colors. Using these two girih modules in a rotational fashion, results in the pattern in this artwork (Figure 13.M). To make an actual tiling for this pattern a craftsman may use the cut Sâzeh tiles that are presented in Figure 13.R.

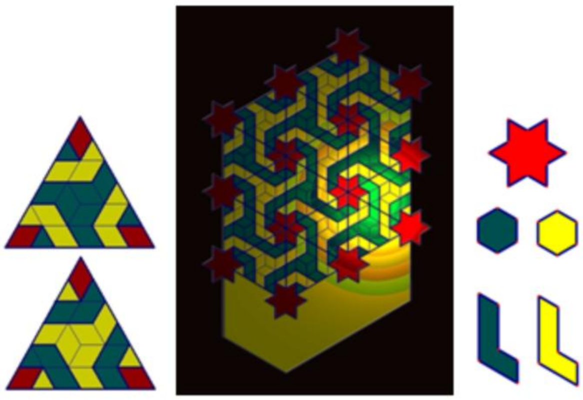
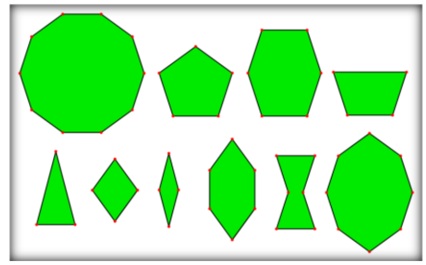


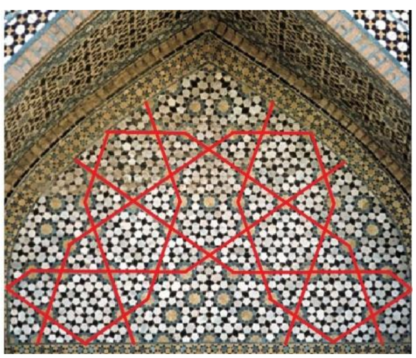
Figure 13: (L) *The two modules used to find the layout of a tessellation,* (M) "*Hope" (December 2008), and* (R) *The sâzeh tiles that were used to create this pattern.*

Figure 14



6. Modularity in Interlocking Star Polygon Mosaic Designs

Jay Bonner explained the polygonal system, *polygons in contact*, which was used in the creation of the patterns in the past (see Figure 6) [2]. Figure 14 illustrates the ten polygons that form the five-fold system. In this method, a craftsman makes an underlying polygonal matrix that is generated from modules in Figure 14 **(**Figure 6.L). Then he uses the midpoints of the polygons to discover new lines that form a new tiling pattern (Figure 6.M). When the process of creation is complete, then the initial polygonal matrix is discarded (Figure 6.R).



Looking at Figure 15 (Darb Imam, Isfahan) Bonner noticed a set of lines that connect the centers of decagrams to form another tessellation with larger composite tiles (these lines have been made bold to be more visible). He used this figure in his treatment of Self-Similarity in the Medieval Persian mosaic design.

Figure 15

An informative book that appeared in recent years about mosaic design and its history is *The Topkapi Scroll* [14]. The book includes all the images on the Topkapi Scroll. The scroll presents 114 images for creating designs. Bonner used the Topkapi Scroll: No. 28 image (Figure 16.L) as another example for the 5-fold Self-Similar Type A: "Pattern 28 in the Topkapi scroll is a 5-fold self-similar Type A design that also depicts the underlying polygonal sub-grid used in the creation of the secondary design…. That this very particular technique was used historically is conformed in the Topkapi scroll. Pattern 28 from this scroll makes use of small red dots to distinguish the underlying polygonal sub-grid of the secondary pattern." [2]

Lu and Steinhardt also noticed these red dotted lines. They proposed that these dotted lines exhibit a new set of tiles, where the black solid lines determine the design on each of these new tiles [11]. They realized that this new set can be used as a set of modules, similar to the modules that were presented in the previous section, but now in more complex forms, for finding new interlocking star polygon patterns. This eliminates the hardship involved in a compass-straightedge construction and, in fact, opens the door for creating more interesting mosaic patterns that are not formed in a usual way. Lu and Steinhardt called this new set *girih tiles.* This is the name that we use in this paper for all the modules that form the layout of a Persian mosaic tiling.

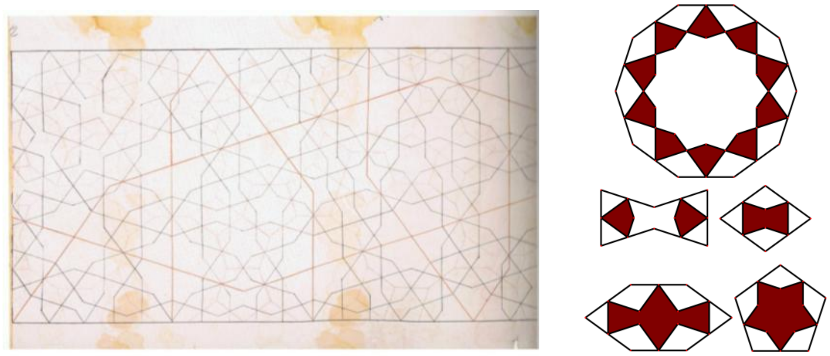


Figure 16: (L) *A pattern from the Topkapi Scroll book,* (R) *The set of girih tiles.*

Using three modules from this set will enable us to construct the four aforementioned tessellations that were constructed using a compass and straightedge. In Figure 17, from left to right, one finds how the composition of three girih modules from the five modules in Figure 16 can generate the tilings in Figure 1, Figure 4, Figure 7, and Figure 11 respectively.

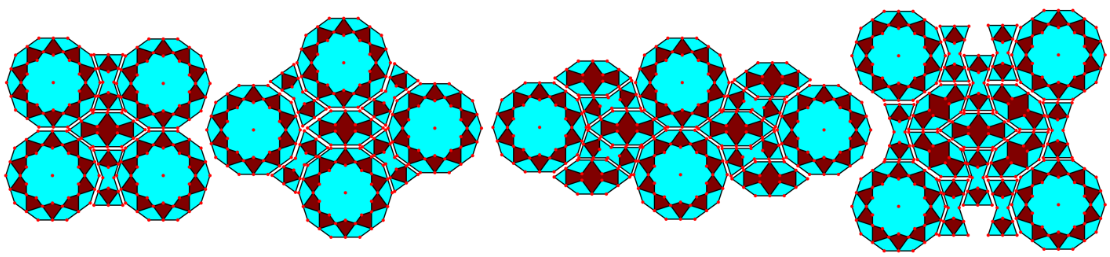


Figure 17

Obviously, the design in Figure 15 is highly complex and much more complicated than the other four tessellations in Figure 17. However, using the girith modules set one can construct it in much easier fashion than using compass-straightedge. Figure 18 shows how one can construct the larger tessellation tiles. Then executing the entire tessellation on a wall for a craftsman is a matter of time.

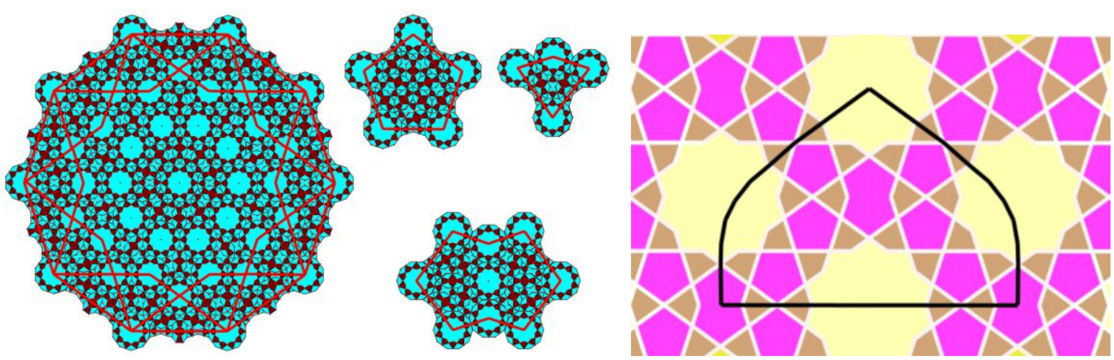


Figure 18

Penrose discovered the two tiles of kite and dart, each designed by some arcs, that only can tessellate non-periodically (no transitional symmetry) (Figure 19.M). These two tiles form a rhombus that is a wing of a five-fold star that can be constructed using the (10, 3) Star Polygon (Figure 19.L). In order to tile properly using kite and dart one should also connects the arcs of the same color on the tiles properly to create continuous curves (closed or open). Based on a set of compounded tiles created by John Conway [8], Lu and Steinhardt proposed three new tiles that resemble three of girih tiles and generates Penrose tilings (Figure 19.R).

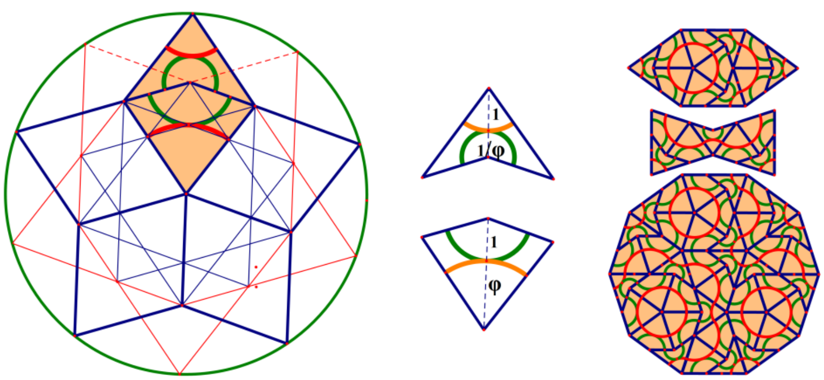


Figure 19

Using this new set in Figure 19.R and replacing them with the girih tiles of Figure 16. R Lu and Steinhardt suggested a possible quasi-periodicity structures in some Persian tilings. Similar quasi-periodic patterns were noticed and analyzed by Makovicky [13] a few years earlier. Moreover, Rigby [15] discovered a way to cover the surface of kites and darts with appropriate Sâzeh tiles to generate various non-periodic interlocking patterns. Nevertheless, the discovery of the girih tiles in Figure 16 was without doubt a remarkable point in the study of mosaic designs.

Note that by using the tiles suggested by Rigby one can make the three girih modules in a way that the tiles tessellate the plane non-periodically (Figure 20).

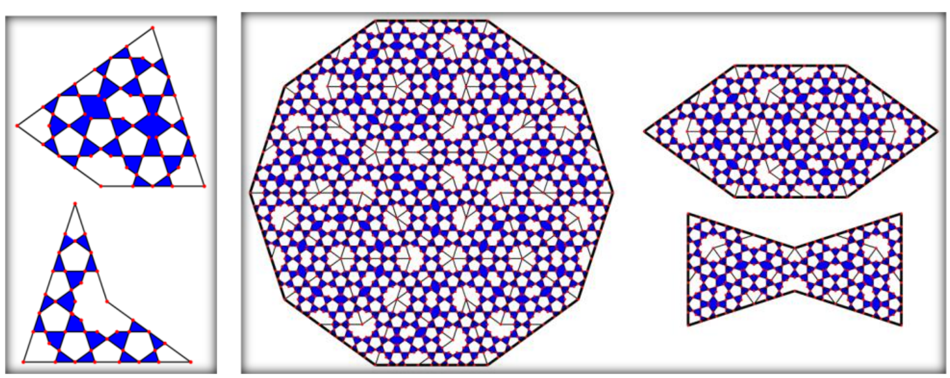


Figure 20

Looking at similar designs as in Figure 16 one may come to this conclusion that the designers of the past were looking for maximum symmetries, especially local and global rotational symmetry, especially 5-fold and 10-fold rotational symmetries, than anything else. So some of their mosaic designs have attracted the modern crystallography researchers, who have found similar patterns.

It is important to note that the Penrose tiles cannot produce a global or local 10-fold rotational symmetry. They only can form 5-fold symmetries. There are uncountably many Penrose tilings. None of them have global 5-fold rotational symmetry but two. The two tessellations that have global 5-fold rotational symmetry are called “Sun” and “Star”. Figure21. L is the Sun tessellation. Figure 21.R is the same tessellation covering with the Rigby tiles. The central circular part of the tessellation in Figure 21.R, which includes the first series of decagrams that are equidistance from the center, resembles the decagonal girih tile in Figure 18 L. Nevertheless, a bigger portion of this Sun only holds 5-fold rotational symmetry but the decagon in Figure 18L has 10-fold rotational symmetry.

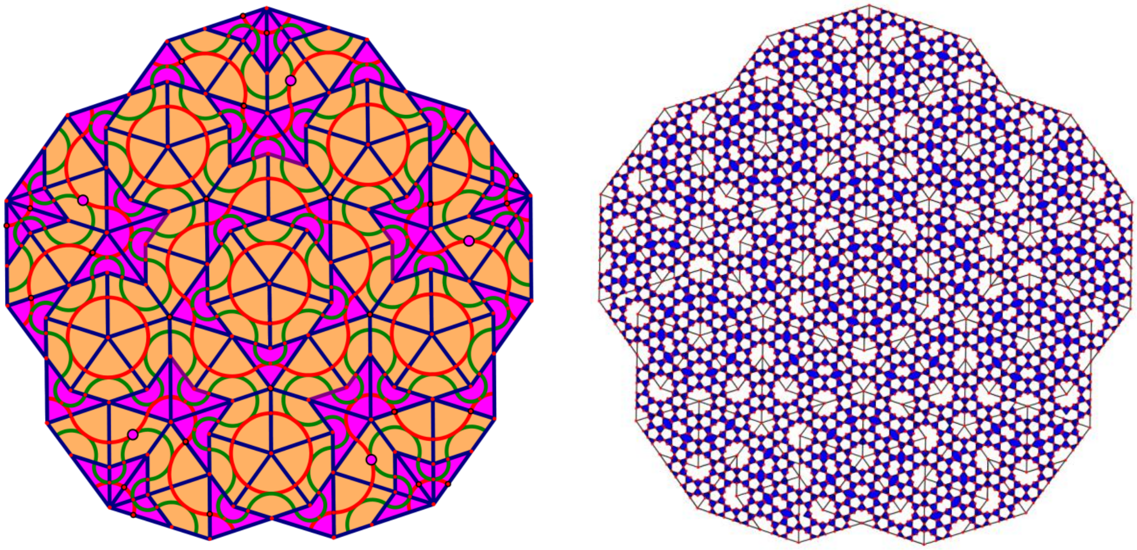


Figure 21: Penrose Sun and the Girih Sun

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