The *Topkapı Scroll's* Thirteen-Pointed Star Polygon Design

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Abstract

This paper will explore the *Topkapı Scroll* sketch known as Catalog Number 30*,* the only *repeat unit,* that, when replicated using symmetry operations, creates an overall pattern consisting of "nearly regular" thirteen-pointed stars, regular sixteen-pointed stars as well as irregularly-shaped pentagonal stars. Since there are no known written instructions explaining how this repeat unit was generated, we set out to see if we could recreate this highly unusual design using only the simplest of Euclidean construction techniques that may have been known to, and used by, master builders during medieval times.

Introduction

The *Topkapı Scroll*, a well-preserved 96-foot-long scroll thought to date from the 15^{th} or 16^{th} centuries, contains 114 Islamic architectural and ornamental design sketches, most displaying only a small portion, or *repeat unit,* of an overall Islamic pattern. Usually surrounded by a square or rectangular border, each repeat unit of the *Scroll* may be reflected across its edges or rotated about its vertices to achieve the entire pattern. Since the *Scroll* contains no accompanying explanations on how the repeat units may be generated, it is believed that these sketches most likely "served as an *aide-memoire* for architects and master builders who were already familiar through experience with the coded graphic language used in them" [1, pages 10 - 12]. How, then, did artisans and master builders centuries ago create these elaborate Islamic star polygon designs? Although few written records exist, it most likely did involve "a considerable amount of geometrical knowledge" [2], suggesting that some degree of mathematical literacy may have existed among the master builders, architects and master engineers. Theoretical mathematicians (al-Sijzī, Abū Nasr al-Farabī', Abu'l-Wafā' al-Buzjani, Al-Kashi, Umar al-Khayyami, and Abu Bakr al-Khalid al-Tajir al-Rasadi, among others) developed and wrote about construction techniques useful to artisans interested in creating geometric ornamentation [3, 4]. Manuals were written as a result of the meetings between these two groups, including one by al-Buzjani entitled, *Kitāb fīmā yahtāju ilayhi al-sāni' min a'māl al-handasa* (About that which the artisan needs to know of geometric constructions) which provided simplified instructions on how to perform basic Euclidean constructions using the traditional drafting tools of the medieval period – a compass, straightedge, and set square – for (among other things)

> the construction of a right angle; the bisection of a square or circle; drawing a line parallel to, perpendicular to, or at a certain angle to a given line; determining the center of a circle or its arc; dividing the circumference of a circle into equal arcs… From these general problems al-Buzjani moved on to the construction of regular polygons inscribed in circles, other constructions involving circles and arcs, and the constructions of polygonal figures inscribed in various figures. The circle is used in al-Buzjani's treatise to generate all of the regular polygons in a plane [1, p. 138].

These geometric constructions could well have formed the basis for creating many of the geometric Islamic patterns of the time.

Topkapı Scroll, Catalog Number 30

The repeat unit sketch known as Catalog Number 30 (or CN30 for short) of the *Topkapı Scroll,* generates a pattern consisting of "nearly-regular" thirteen-pointed star polygons, and it the only repeat unit of the *Scroll* to do so. The original repeat unit was drawn in black ink on cream-colored rag paper [1, page 29], a facsimile of which is reproduced in [1, page 301]. The author's reproduction of CN30, made by tracing the image and producing its copy using the *Geometer's Sketchpad* software program [5], is shown in **Figure 1**. (Note that the right edge of the sketch, where it is pasted to another section of the *Scroll*, is missing, but the overall design may be inferred due to the symmetry of the pattern). The CN30 repeat unit contains two types of *major stars*, that is, those stars that are the largest in size and command the most attention in the design; in this case the major stars are the thirteen-pointed and sixteen-pointed star polygons. In addition, the repeat unit has one type of *minor star*, the irregularly-shaped pentagonal stars, whose main purpose is just to fill in the interstitial space between the major stars. Two halves of the thirteen-pointed star polygons appear along the right vertical edge and the bottom edge of the square, while a quarter of the sixteen-pointed star polygon is centered on the square's upper left vertex. Between the two half-thirteen stars are two irregularly-shaped pentagonal stars; and between the half-thirteen stars and the quarter sixteen-star are four arrow-like polygons and a tri-lobed non-convex polygon. Note that the CN30 repeat unit has mirror symmetry across the diagonal that runs from the upper left to the lower right corners of the square, which is useful in the construction of the pattern.

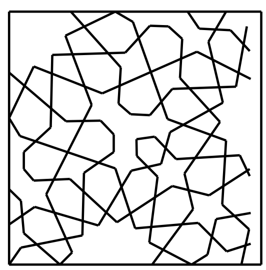

Figure 1. *Author's reconstruction of the CN30 repeat unit produced using the Geometer's Sketchpad*

Besides being the only repeat unit in the *Topkapı Scroll* that generates a thirteen-pointed star polygon design, CN30 is extraordinary for two reasons. First, in the vast majority of geometric Islamic star designs, the major stars usually have an even number of points. And, although CN30 does have even sixteen-stars, it also has major stars with an odd number of points (the thirteen-stars). Second, the major stars usually appear centered at either the corner vertices (as the quarter sixteen-pointed star is) or at the midpoints of the edges of the repeat unit. But for CN30, the centers of the thirteen-stars are not located in either of these standard positions.

Since meetings were held between mathematicians and craftsmen starting in the 10th century, and "practical geometry" manuals providing instructions on how to perform basic geometry constructions were written, it seems plausible that some master builders may have been capable of using geometric construction techniques to create new, and already known, patterns. With this in mind, we set out to see if we could recreate the exact proportion and placement of the star polygons in the highly unusual CN30 repeat unit, using only simple Euclidean construction techniques. Since there are no known written records on how the repeat unit was actually achieved, our reconstruction may be considered an intellectual exercise to determine if it is possible to find a plausible, rather straightforward "point-joining" construction using only the tools available to medieval craftsmen.

Construction of the CN30 Repeat Unit

In this section, we outline a method to construct the CN30 repeat unit square and the polygons that comprise the design within it. First, draw a line segment of any length and construct six congruent circles with their centers on the segment, as shown in **Figure 2a**. This may easily be accomplished by constructing the first circle of any size radius with its center on the segment (in bold, as the third circle from the left, below) and then using the two points of intersection of this circle with the segment as the centers for two additional, adjacent and congruent circles – one on either side that go through the first circle's center point (shown as the second and fourth circles from the left, below). Continue in this manner until there are a total of six congruent circles whose centers are on the segment (the dashed circles are the last three to be added). Note that the segment has now been partitioned into seven congruent parts (formed by the radii of the circles). Erasing the circles and all the points except the center point of the first circle and the two endpoints yields **Figure 2b**, which is partitioned into a three-unit section (on the left) and a four-unit section (on the right).

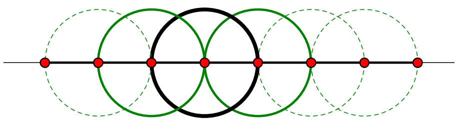

Figure 2a.

Figure 2b.

Before continuing with the construction, we note that one of the most straightforward methods for creating star polygons is to initially construct a p-gon (a polygon with p sides), or alternately, partition a circle into p congruent arcs, and then draw in the corresponding regular p-pointed star by methodically joining the qth vertices of the p-gon (or the qth point along the circle) with line segments. A figure formed in this way is designated as a $\{p/q\}$ star polygon, where p and q are relatively prime positive integers, with $q < p/2$. After examining the portion of the sixteen-star found in the CN30 repeat unit (which is reproduced in **Figure 3a** and also shows the area of one point shaded), we find that the exact proportion of the star may be produced by constructing a {64/23} star (that is, a 64-pointed star where every 23rd point is connected), and then erasing all of the star's points except every fourth one (the dotted line segments in **Figure 3b** are the ones to be erased). For comparison purposes, portions of a {32/11} star, with every other point erased (indicated by dotted line segments), a {16/5} star and a {16/6} star, are shown in **Figures 3c – 3e,** respectively, but none of these match the sixteen-star in the CN30 repeat unit.

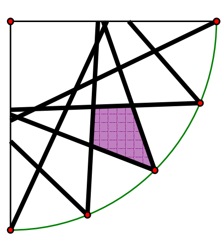

Figure 3a. *CN30*

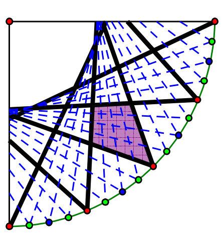

Figure 3b. *{64/23}*

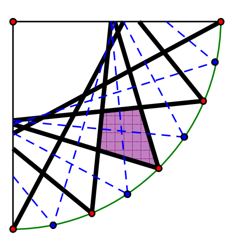

Figure 3c. *{32/11}*

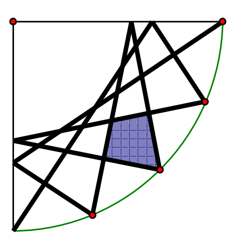

Figure 3d. *{16/5}*

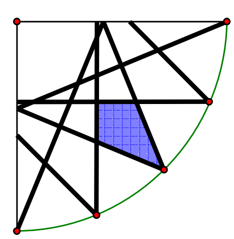

Figure 3e. *{16/6}*

To continue, we construct a square with sides congruent to the segment and a circle centered on the upper left vertex and through the existing point on the upper edge of the square (that was the center for the first circle, the third one from the left shown in **Figure 2**). Bisect the upper left angle of the square, and all the subsequently formed angles, until there are fifteen bisectors yielding fifteen points of intersection on the circle, as shown in **Figure 4a** on the following page. Additional points along the remaining three-fourths of the circle may easily be found until there are a total of 64 equi-spaced points (or 64 congruent arcs) on the circle as shown in **Figure 4b** (after the bisectors have been erased). To produce a quarter of the sixteen-star polygon, connect the point on the circle that intersects the left edge of the square with the two points 23 congruent arc lengths away, found by moving clockwise and also counterclockwise from this position. Move four arc lengths counterclockwise to a new point and connect with segments this new point with the two points 23 arc lengths away, and so on. We repeat this procedure until reaching the point on the circle that intersects the upper edge of the square, producing a quarter of a {64/23} star polygon (but with all of the points erased except every fourth one, as shown in **Figure 4c**. The points of intersection of these segments may be used to construct two smaller concentric circles within the original circle (as shown in **Figure 4d**). Extend the two segments that lie outside the square and that pass through the first and the last points on the circle until they intersect the right and lower edges of the square, as shown in **Figure 4e**. Erasing the points, segments, and circular arcs that lie outside, and other unneeded segments inside, the square yields a quarter of the sixteen-star shown in **Figure 4f**.

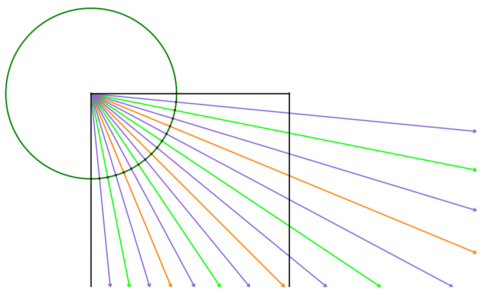
Figure 4a.

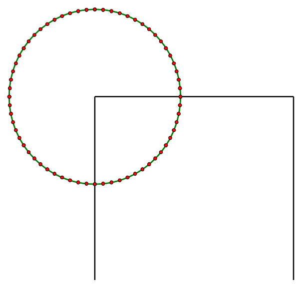
Figure 4b.

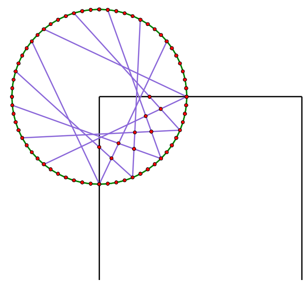
Figure 4c.

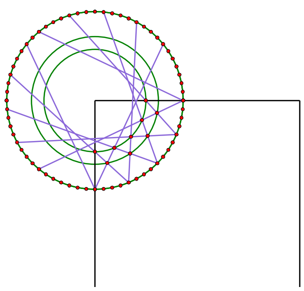
Figure 4d.

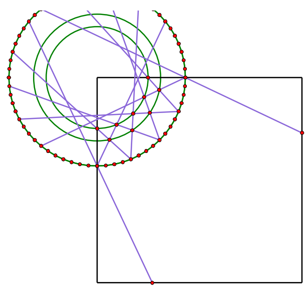
Figure 4e.

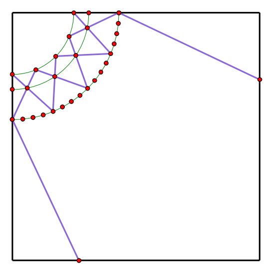
Figure 4f.

Erase the two smallest circular arcs inside the square and extend existing segments until they intersect the edges of the square as shown in **Figure 5a**. Then construct a diagonal and two other segments between existing points until they intersect the edges of the square, as shown in **Figure 5b**. Now bisect the angle formed by the lower edge of the square and the highlighted segment emanating from the upper left corner. Construct a semicircle about the vertex of this angle, with the radius defined by the intersection of the angle bisector with an existing segment as shown in **Figure 5c**. Bisect the angle formed by that same highlighted segment emanating from the upper left corner and the newly constructed segment. Construct a second larger semicircle (concentric with the first) through the point of intersection formed by the second angle bisector and this same segment (see **Figure 5d**).

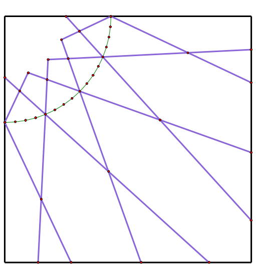
Figure 5a.

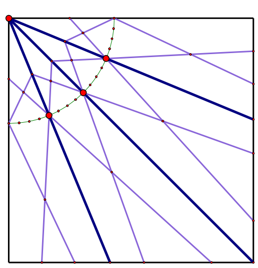
Figure 5b.

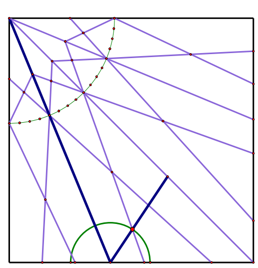
Figure 5c.

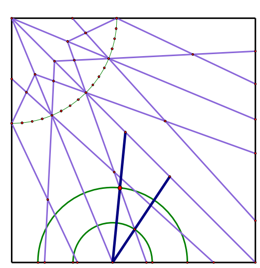
Figure 5d.

Construct two additional segments through existing points as shown in **Figure 5e**. Construct the angle bisector of the angle formed by one of the newly added segments and the lower left edge of the square, as shown in **Figure 5f**. Two additional segments may now be constructed through the points of intersection of these segments and the center of the 13-star on the lower edge of the square, as shown in **Figure 5g**.

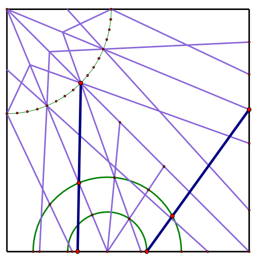

Figure 5e.

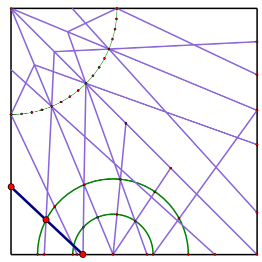

Figure 5f.

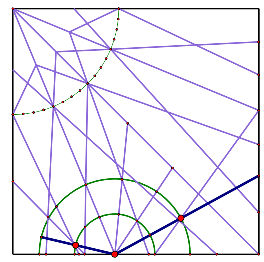

Figure 5g.

We are now able to construct or highlight existing segments that comprise most of the points of the thirteen-star that has its center on the lower edge of the square, as shown in **Figure 5h**. To complete the star, construct a third semicircle concentric with and between the first two to get the necessary point of intersection on the lower edge of the square, allowing for the construction of the needed segments, as shown in **Figure 5i.** Construct a fourth (and the largest) concentric semicircle defined by the highlighted point of intersection, and extend the existing segments comprising the thirteen-star until they meet this semicircle (see **Figure 5j**). Construct a small circle in the lower right corner of the square to produce a point of intersection with the largest semicircle, which allows for the construction of the final segments needed to complete the thirteen-star and part of the rosettes, as shown in **Figure 5k**.

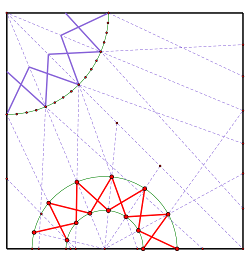

Figure 5h.

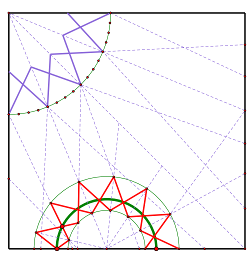

Figure 5i.

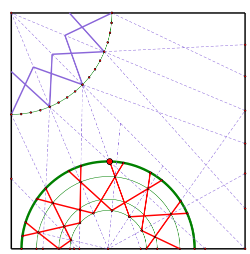

Figure 5j.

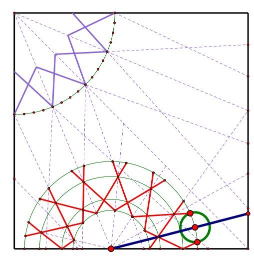

Figure 5k.

In order to declutter the sketch, erase the now unneeded segments and semicircles (**Figure 6a** on the following page). The other thirteen-star (that has its center on the right edge of the square) may be constructed in a similar manner to the first due to the mirror symmetry across the diagonal (**Figure 6b)**. With the portions of the major stars within the square now constructed, all that remains is to fill in the interstitial space with the requisite polygons. To do this, construct intersecting segments through existing points, thus forming a point which may be used to define the radius of an additional semicircle concentric with an existing one shown in **Figure 6c**. Four segments emanating from the center point of the thirteen-star (which has its center on the bottom edge) may now be constructed through existing points and extended until they intersect the semicircle, as shown in **Figure 6d**. Erase the semicircles. The points of intersection just formed on the semicircle may now be connected with line segments to other existing points yielding **Figure 6e**. The same constructions may be produced about the thirteen-star whose center is on the right edge (again due to the symmetry across the diagonal), as shown in **Figure 6f**. Highlight the needed segments, while erasing those no longer needed. The point highlighted in **Figure 6g** is the one now used to define the radius of a second, larger circular arc to be constructed about the center of the sixteen-star in the upper left corner of the square. Extend existing segments of the sixteen-star until they meet the arc, and then join these and other points to additional points, as shown in **Figure 6h**.

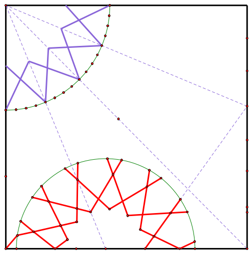

Figure 6a.

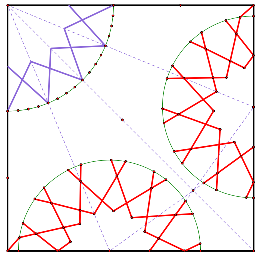

Figure 6b.

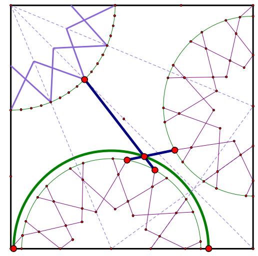

Figure 6c.

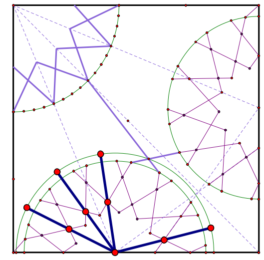

Figure 6d.

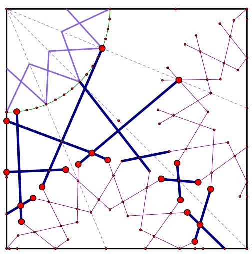

Figure 6e.

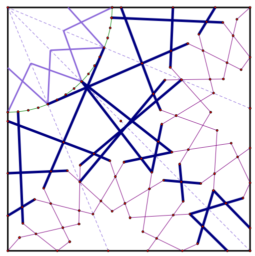

Figure 6f.

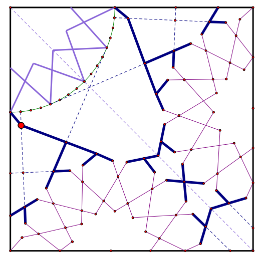

Figure 6g.

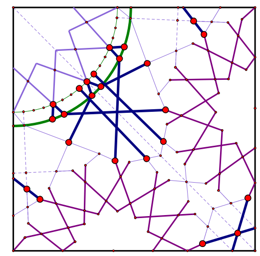

Figure 6h.

Erase this second, larger arc and a few unneeded segments to get **Figure 7a**. The construction of two sets of concentric circles yields the highlighted points of intersection (**Figure 7b**) needed to construct the segments shown in **Figure 7c**. Erasing these circles and constructing four additional, smaller circles produces the highlighted points and additional segments shown in **Figure 7d.**

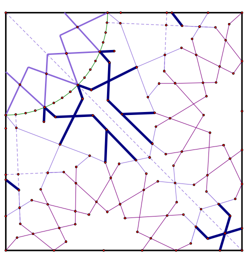

Figure 7a.

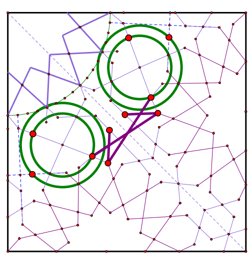

Figure 7b.

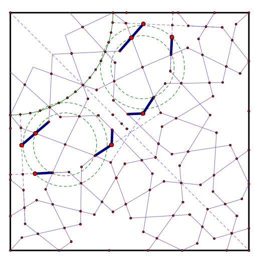

Figure 7c.

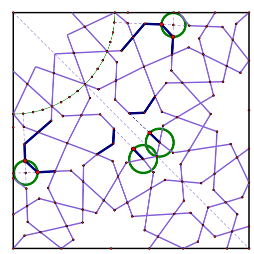

Figure 7d.

Erase the circular arc and points about the sixteen-star in the upper left corner of the square, and the four small circles and other now unneeded line segments. Construct two additional segments between existing points and a small circle through the four end points of these segments to generate the last point needed in our sketch, as shown in **Figure 7e**. The completed repeat unit sketch of CN30 is shown in **Figure 7f**. The repeat unit with the points erased and the line segments of the stars colored so as to distinguish them from each other and the connecting polygons is shown in **Figure 7g**.

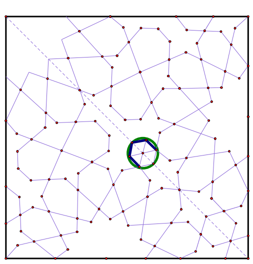

Figure 7e.

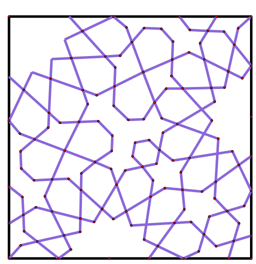

Figure 7f.

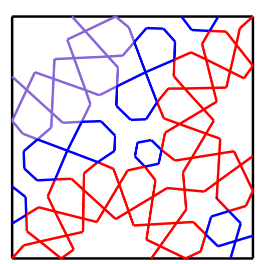

Figure 7g.

Four colored copies of the repeat unit, replicated by reflection across the square's edges (shown in **Figure 8a)** illustrates that the four thirteen-stars are linked, with a point of one star's rosette touching similar points on its two adjacent thirteen-star neighbors, pairwise. The sixteen-stars and their rosettes do not touch any other sixteen- nor thirteen-star polygons and their rosettes. An alternate view of the four colored copies of the repeat unit, where a single sixteen-star is at the center of the square surrounded by eight half thirteen-stars, is shown in **Figure 8b.**

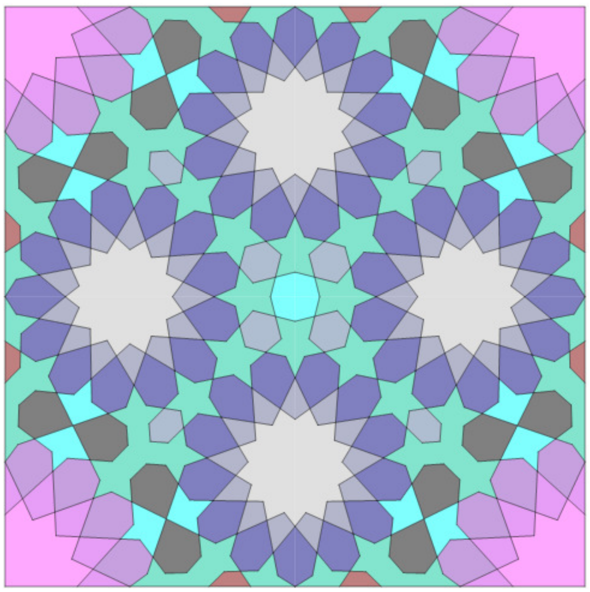

Figure 8a.

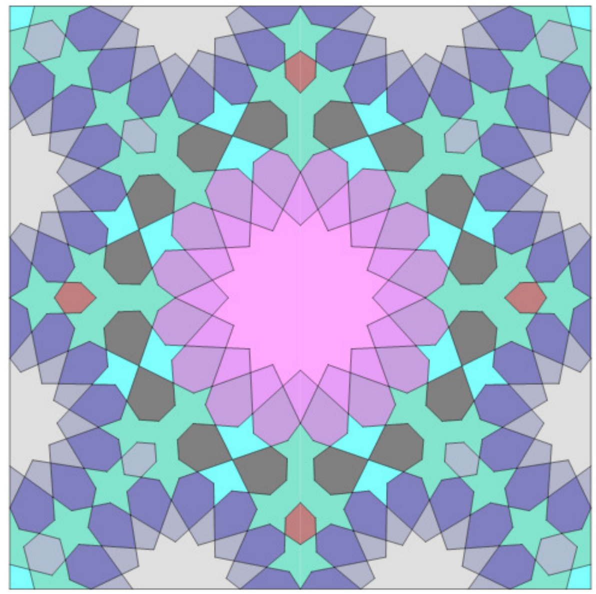

Figure 8b.

Discussion

Given the discussions on "practical geometry" that took place between mathematicians and craftsmen starting in the 10th century, it seems plausible that some geometric construction techniques may have been used to generate Islamic star patterns and their repeat units during medieval times. Because there are no known written instructions explaining how the CN30 repeat unit was generated, we set out to see if it was possible to recreate the exact proportion and placement of the star polygons in the highly unusual design using only the simplest of Euclidean construction techniques. We succeeded at finding a possible, rather straightforward "point-joining" compass-and-straightedge construction, requiring nothing more complicated than the construction of angle bisectors. However, as previously noted, there is no historical evidence that the particular sequence of steps described here is precisely what may have been used for the creation of CN30. All that may be concluded is that they do produce the pattern in a rather simple, straightforward way making it a possible and plausible construction.

The extensions of the line segments comprising the regular sixteen-star star polygon form the basis for the construction of the segments comprising what turn out to be "nearly regular" {13/5} star polygons. The theoretical central angle measure between adjacent points of a *regular* thirteen-star would be ~27.69°. The actual angle measures for the points of this constructed thirteen-star (as measured by the *Geometer's Sketchpad* software) are shown in **Figure 9** on the following page (note that the angles for the primed points are the same for their unprimed counterparts), and range from being nearly exact (-0.2 % off) to being ~5.2 % off the theoretical central angle measure (see **Table 1** on the following page).

The CN30 design is the only idealized, recorded pattern in the *Topkapı Scroll* containing thirteen-pointed star polygons, and there are no other such recorded pattern templates of which the author is aware. There are none to be found in the *Tashkent Scrolls,* which consist of fragments of architectural sketches attributed to an Uzbek master builder or a guild of architects practicing in 16th century Bukhara [1, p. 7]. Nor are there any extant examples of thirteen-pointed star polygon patterns in two other rich published sources of Arabic geometric designs: Bourgoin's *Arabic Geometrical Pattern and Design* [6], a manual of over 200 Islamic patterns, and David Wade's collection of over 4000 images [7]. There is, however, a thirteen- and eleven-pointed star polygon design found on an exterior panel of the *Tomb of Mu'mina Khatun*, built in 1186 CE in Nakhchivan, Azerbaijan. But, with the nearly regular {11/4} star polygons appearing vertically through the center of the panel surrounded by pentagonal stars and four nearly regular {13/5} star polygons in two alternate configurations, it has a very different structure from that of CN30. See [8] for the thirteen- and eleven-pointed star polygon design in Nakhchivan.

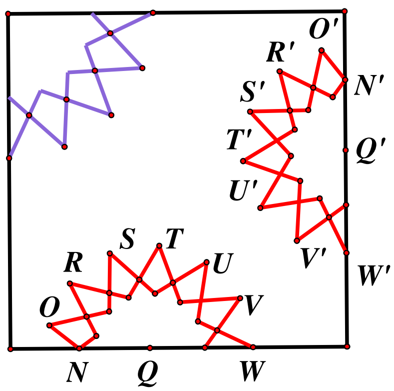

Figure 9.

	angle measures (in degrees)	difference between actual & theoretical angle measure	difference in percent
$2m\angle OQN = 2m\angle O'Q'N' =$	26.26*	-1.43	-5.2
$m\angle OQR = m\angle O'Q'R' =$	26.24	-1.45	-5.2
$m\angle RQS = m\angle R'Q'S' =$	27.64	-0.05	-0.2
$m\angle SQT = m\angle S'Q'T' =$	28.12	0.43	1.6
$m\angle TQU = m\angle T'Q'U' =$	28.12	0.43	1.6
$m\angle UQV = m\angle U'Q'V' =$	28.12	0.43	1.6
$m\angle VQW = m\angle V'Q'W' =$	28.61	0.92	3.3

Table 1. **26.26° was obtained by doubling the 13.13° angle measure of $\angle OQN$*

Acknowledgements

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