Playing with the Platonics: a new class of polyhedra

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Abstract

Intrigued by the impossibility of making a closed loop of face-to-face connected regular tetrahedra, I wondered how adjustments to the polyhedron could make it loop-able. As a result I have defined a method to construct a whole class of new polyhedra based on the Platonic solids. By exploring this class I found several examples of polyhedra that do make closed loops possible, and sometimes it is possible to build 3D lattices or other regular 3D structures with them. This project was however not a complete analyses of all possibilities, but merely a short study.

Introduction

It has been proven (see [1] and [2]) that by connecting regular tetrahedra face-by-face, it is impossible to make a connected and closed loop, also known as a toroidal polyhedron. If for instance five tetrahedra are connected in a ring a small gap is left (Fig. 1), and also for longer strings (like Fig. 2) one can at most find near misses, but never closed loops. Making a regular helix however is easy (see Fig. 3 and [3]).

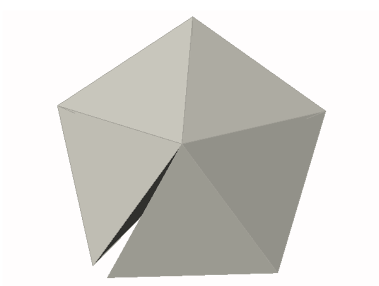

Figure 1: *5 Tetrahedra*

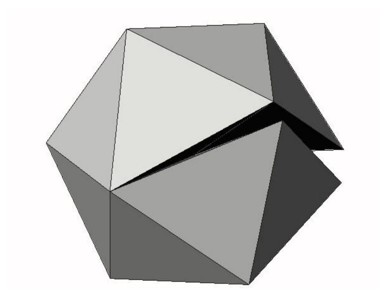

Figure 2: *8 Tetrahedra*

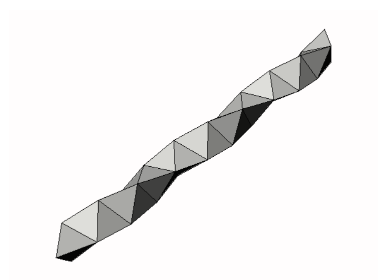

Figure 3: *Helix of tetrahedra*

Figure 4: *A near miss loop for the snub-cube, seen from various viewpoints*

Another example of a "non-loop-able" polyhedron is the snub cube (if only one of both enantiomorphs is allowed). Figure 4 shows three different views of a "near miss loop", a constellation of 45 snub cubes. This construction led to the idea that by rotating the square faces of these snub cubes just a little, it would be possible to change it into a closed loop. For this the rotation axis should pass though the center of the square face and the center of the polyhedron (and the rotation angle be only approx. 0.1039°). The triangles will no longer be regular, but this is impossible to see by the eye. But now the polyhedron can form a loop (of 44 pieces)! This slightly changed snub cube can also be constructed as follows: start with a cube, scale the 6 square faces down towards their centers (by the right factor), rotate them around the axes mentioned above (by 180/11 ≈ 16.3636°) and then take the convex hull of the result (which adds the connecting triangles).

General Construction Method

This method can be extended in order to create a wide range of possible polyhedra. I have used the following procedure (describing also its variations and limitations), of which Figure 5 shows an example:

1. The "**base**" polyhedron to start with is a Platonic solid (Tetra, Cube, Octa, Dodeca or Icosa). So there are 4, 6, 8, 12 or 20 original faces (call this number **p**).
2. The new face to "fit in" each original face, can be any **regular n-sided polygon**. It should be coplanar with the original face and also have the same center.
3. There is a **scaling factor s**, which influences the radius of the new polygon. There is a maximum radius where the new vertices touch the edges of the original face.
4. It can be **rotated by an angle r**, with the rotation-axis through the face-center and the body-center.
5. After the first new face is created in an original face, there are several ways to copy it (in my Rhinoceros/Grasshopper computer program) to the other original faces. Each step can be done by rotation or by reflection. In the first case, the spin of both faces is the same, in the latter case they have opposite spin. I defined some different **configurations** (=combinations of all these choices) for each platonic solid. See next page for more explanation on this subject.
6. At this moment there are **p** new faces, I call them the "primary faces". By finding the convex hull of vertices of these faces, we can find all the secondary faces, connecting the primary ones. Together they define the new polyhedron. Connecting more polyhedra to find possible loops should only be done by joining these primary faces. Here chiral opposites of the polyhedrons are excluded.

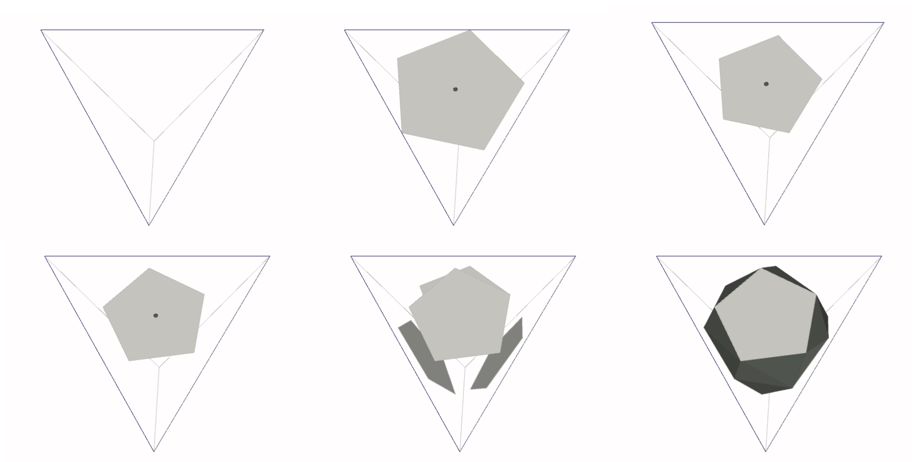

Figure 5: *Example of the construction method, with: base= tetra (so p=4), n=5, s=0.8, r=20°, conf=1 (all primary faces have the same spin).*

The configurations mentioned in step 5 can be seen as the (arbitrary) choices for each face of the direction of the rotation of the primary face (the spin), and for the point from where to rotate (the zero angle reference points). Figure 6 shows all the 14 different configurations that I have analyzed. Those with indices 1 have faces with only one kind of spin, the others have both spins in equal numbers.

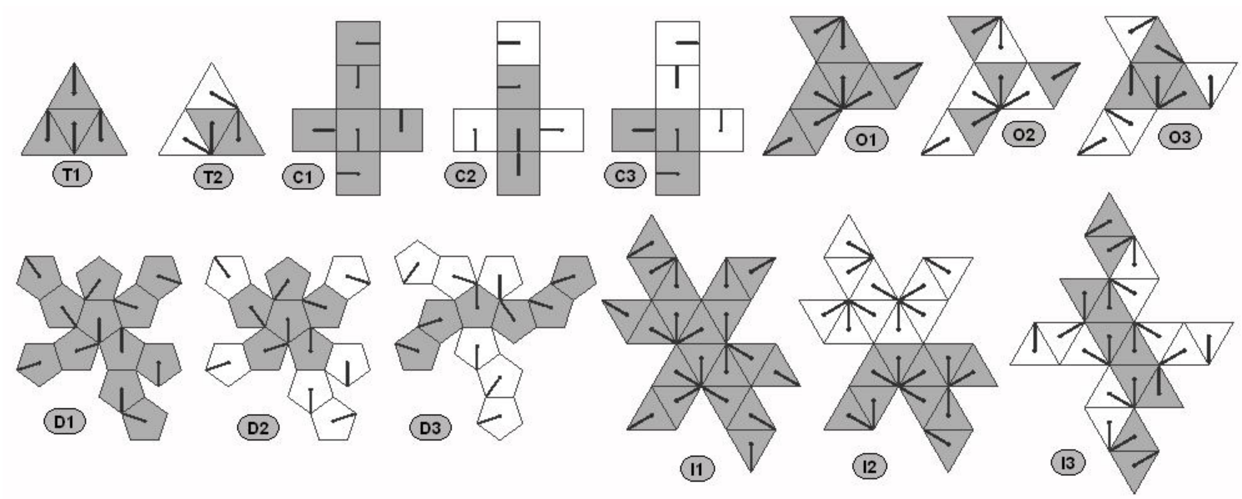

Figure 6: *Unfolded nets of the 14 different configurations that were used. The two colors of the faces indicate opposite spins, and the dark lines mark the zero angle reference points.*

Some Results

All the polyhedra that can be created this way (this class forms an infinite set!) have the property that its vertices all have the same distance to the body-center. As the polyhedra are connected via their primary faces, the path connecting all the (body-centers of the) polyhedra is restricted. Between adjacent edges in this path only certain angles are possible (caused by the dihedral angles of the Platonic solids). I call these the connecting angles. Table 1 shows all possibilities, and one can see that the Icosa includes the Octa, which includes the Tetra. The Cube and the Dodeca have different sets of connecting angles.

	41,810	**63,435**	**70,529**	**90,000**	**109,471**	**116,565**	**138,190**	**180,000**
Tetra					X			
Cube				X				X
Octa			X		X			X
Dodeca		X				X		X
Icosa	X		X		X		X	X

Table 1*: All the possible connecting angles (in °) for the 5 Platonic solids*

The polyhedra are somewhat similar to the Symmetrohedra described in [4]. A comparison teaches that:
- I only use 1 of the (face or vertex) symmetries at a time, and never 2 of them together.
- Here **n** does not have to be a multiple of the original axis degree, and angle **r** is not restricted .
- Only the following Symmetrohedra can be made with my method: G(z,1,*,α), G(1,z,*,α), G(z,2,*,e) and G(2,z,*,e), for all integers z ≥ 1. And the two snub Archimedean solids are also included.

So all the 13 Archimedean solids can be constructed with my method. Table 2 shows which base polyhedra (with **p** faces) and **n**-gons can be used for this. The *'s mean that in these cases the primary faces touch by their vertices, so that the total number of vertices of the polyhedron equals **n**·**p**/2, instead of **n**·**p** in all other cases. Nice animations have been made showing for each base polyhedron (the Platonic solid) how it can morph to all its corresponding Archimedean solids and back to its self.

			<<< BASE POLYHEDRON >>>				
Archimedean solid	**Vert.Conf.**	**# of Vertices**	Tetra 4 faces	Cube 6 faces	Octa 8 faces	Dodeca 12 faces	Icosa 20 faces
TruncTetra	3.6.6	12	3-gon or 6-gon*				
CubOcta	3.4.3.4	12	3-gon	4-gon *	3-gon *		
TruncOcta	4.6.6	24	6-gon	4-gon	6-gon *		
SnubCube	3.3.3.3.4	24		4-gon	3-gon		
RhombCubOcta	3.4.4.4	24		4-gon	3-gon		
TruncCube	3.8.8	24		8-gon *	3-gon		
TruncCubOcta	4.6.8	48		8-gon	6-gon		
IcosiDodeca	3.5.3.5	30				5-gon *	3-gon *
TruncDodeca	3.10.10	60				10-gon *	3-gon
SnubDodeca	3.3.3.3.5	60				5-gon	3-gon
RhombIcosiDodeca	3.4.5.4	60				5-gon	3-gon
TruncIcosa	5.6.6	60				5-gon	6-gon *
TruncIcosiDodeca	4.6.10	120				10-gon	6-gon

Table 2: *Table of Archimedean solids, and the* ***base*** *polyhedra and* ***n****-gons needed to construct them*

There appear to be a lot of solutions where the polyhedra form a loop (a closed ring, or toroidal polyhedron, see [5]), but still they are quite exceptional and hard to find, because there are many possible combinations to search through. Also it is easy to make mistakes, as Figure 7 shows.

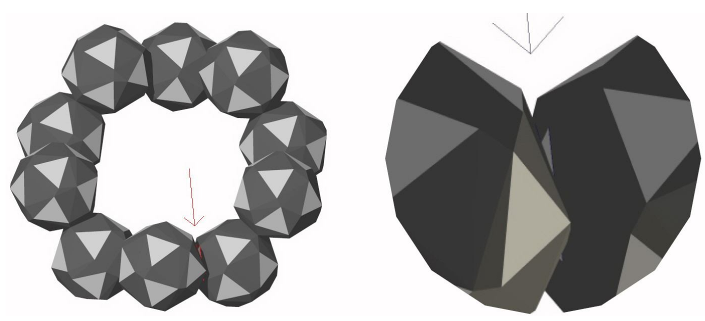

Figure 7: *It is easy to make mistakes: this ring does not close correctly because of a twist ≠ 2π/n. The arrow indicates the misfit, and this is shown in more detail at the right side.*

It was a nice 3D puzzle however to find solutions, Figure 8 shows some of them.

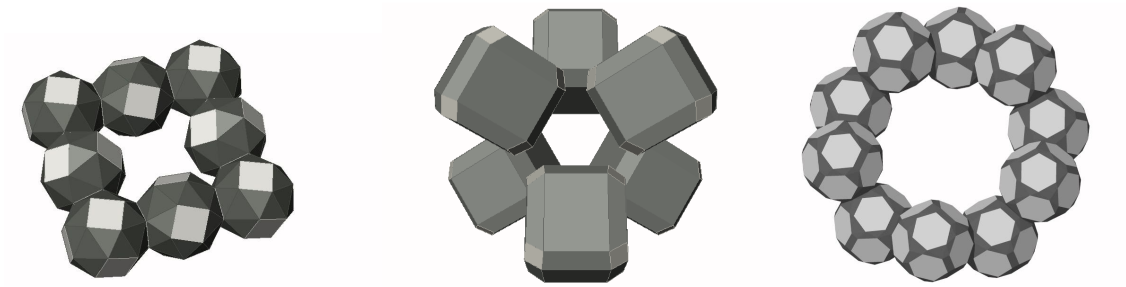

Figure 8: *Some examples of loops (with base of Cube, Octa and Dodeca respectively)*

Some of these solutions (this is even more exceptional) have the additional property that they are valid for every rotation angle **r**. In these cases nice animations can be made showing a closed loop of constantly morphing polyhedra. For this it is required that only faces of opposite spin meet. Three stills of an example of such a rotating loop are shown in Figure 9. Sometimes loops can be found for any $n \geq 3$ (Figure 10), or the number of polyhedra in the loop can be varied (Figure 11).

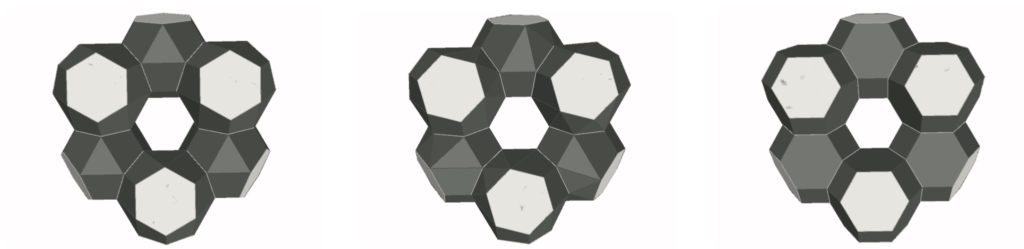

Figure 9: *Three stills from an animation of a rotating loop.*

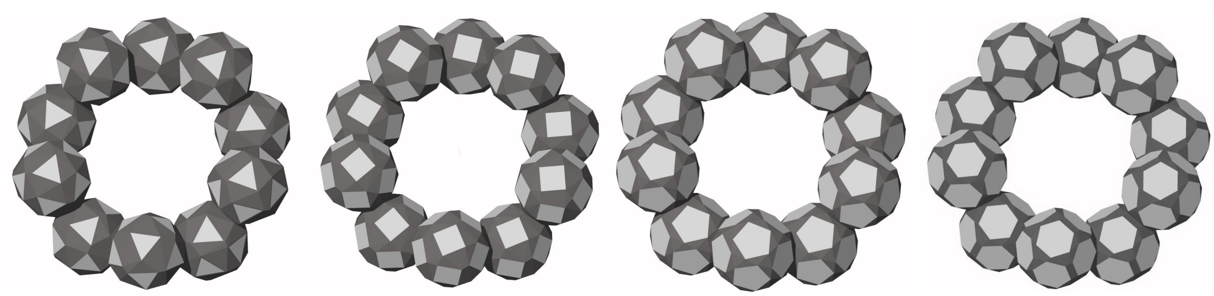

Figure 10: *This loop of 10 Dodecas can close for every $n \geq 3$ (shown here up to n=6).*

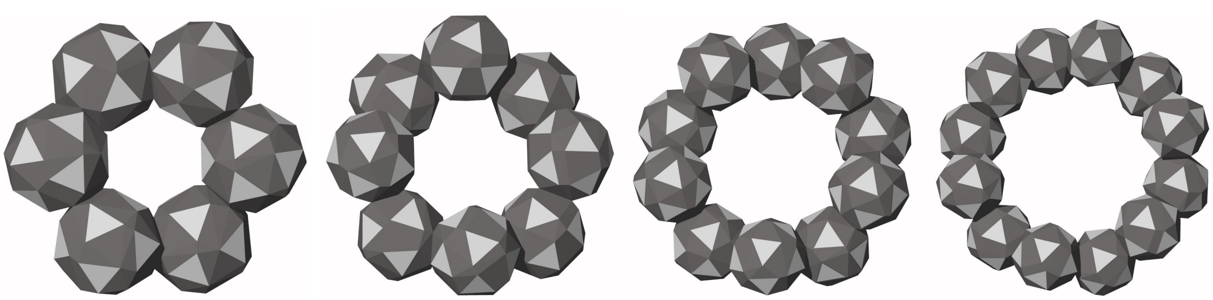

Figure 11: *Loops of 6, 8, 10 and 12 Dodecas, all with n=3 and connecting angle 116.565°. The ring of 6 is unlike the others not a regular skew polygon, but a 6-gon with 2 fold symmetry.*

Sometimes bigger agglomerates of polyhedra can be built, like 3D-lattices (see Figure 12) or so called second order polyhedra. A lot of these solutions have been found earlier, see for instance [5] and [6].

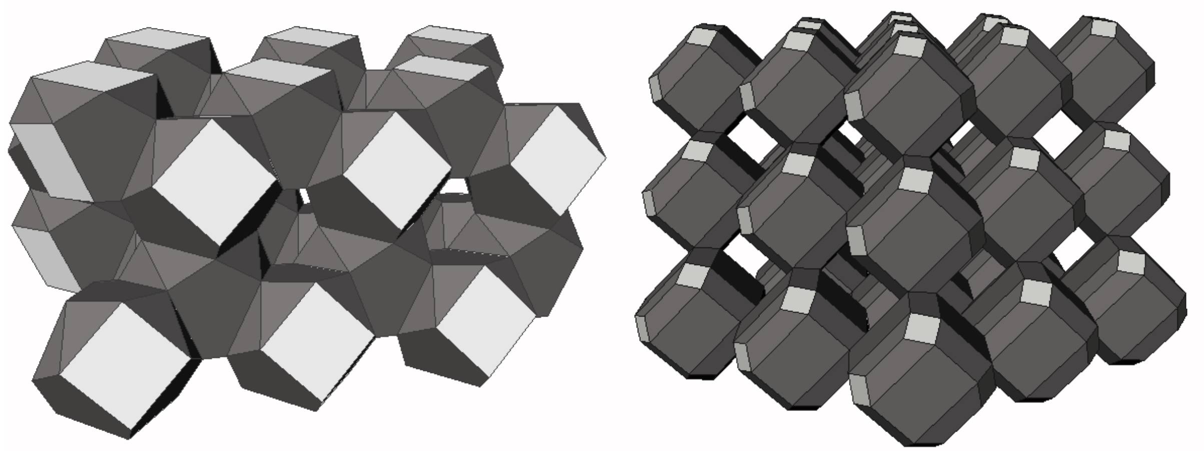

Figure 12: *Examples of (parts of infinite) lattices.*
Left: 24 Tetras with n=4, s=0.7385, r=0°, conf=1. Right: 27 Octas, with n=4, s=0.4, r=0°, conf=2.

Conclusion

Although this research did not lead to a complete analysis of the loop-ability of polyhedra, it gave some great insights into this subject. I have only tried to give some examples of the possibilities in this paper.
Since I was not aware of all the mentioned references until receiving the first comments of the editors, some of the work I have done was merely a rediscovery of already known things.
I hope that the method explained in this paper to construct a whole class of polyhedra out of (in this case) the Platonic solids, will intrigue other people too, and that this will lead to further explorations. I think that a lot of the constructions might make nice sculptures, and there might be possible links, for instance, to architecture, chemistry, and crystallography too.

References

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