A Portrait of a Quadrilateral Group

Jay Zimmerman
Mathematics Department
Towson University
8000 York Road
Towson, MD 21252, USA
E-mail: jzimmerman@towson.edu

1. Abstract

This paper looks at representing a group G of order 16 as a group of transformations of a compact surface of genus three. This is the surface of smallest genus upon which G can be so represented. The action is realized as a quotient of a full quadrilateral group and so all regions have four edges. This group is also NOT the group of automorphisms of a regular map. There are significant differences between this portrait and previous ones.

2. Introduction

The first portrait of a group was given in Burnside [1]. Burnside constructed a portrait of the cyclic group with n elements and the free group, F_n, on n generators in the Euclidean plane. More precisely, since the one to one plane transformations that Burnside used were inversions in a circle, they should be thought of as transformations of the Riemann sphere. The relationship between the circles used determines the resulting group. The construction of the free group, F_2, can be most easily drawn in the hyperbolic plane (see Burnside [1], page 379). By identifying regions on the free group, Burnside could construct a portrait of a finite group. This portrait exists on a compact surface that may no longer be the Riemann sphere. Burnside (see [1], page 396) drew a portrait of the quaternion group of order 8 on a surface of genus 2. The paper, Portraits of Groups [4] drew a portrait of the dicyclic group of order 12 on a surface of genus 2 and a portrait of the quasiabelian group of order 16 on a surface of genus 3. The paper, Portraits of Groups II, Orientation Reversing Actions [5] drew a portrait of a group of order 48, which contained elements that reversed the orientation of its regions. Finally, the paper, A Group Portrait on a Surface of Genus Five [6] represents a group of order 32 on a surface of genus 5. All of these groups are generated by two elements. Since their orientation preserving elements are all quotients of triangle groups of a certain type, their underlying graph is a reflexible regular map. The purpose of this paper is to construct the portrait of a group whose graph is not a reflexible regular map. The group in question will have rank 3. The methods used in this paper are the same as in the previous papers, especially [6].

3. The Group and its Properties

The group that we will use is $G^{+} = \langle 2,2,2\rangle_2$ in the notation of Coxeter and Moser [2] and it is group SmallGroup(16, 13) in the Magma Library. This group has genus 3 and it is a quotient of the Quadrilateral group with presentation $\langle R,S,T,U \mid R^2 = S^2 = T^2 = U^4 = RSTU = 1\rangle = Q(2,2,2,4)$.
The group G is $\langle r,s,t,u \mid r^2 = s^2 = t^2 = u^4 = rstu = (rs)^4 = [r,t] = [s,t] = 1, t^2 = (rs)^2\rangle$.

The portraits of groups given in [1], [4] and [6] are really portraits of a quotient of a full triangle group where the black regions should be labeled with the orientation reversing elements. Therefore, these portraits can be thought of as portraits of extensions of degree 2 of the orientation preserving subgroups, namely the quaternion group of order 8 [1], the dicyclic group of order 12 [4], the quasiabelian group of order 16 [4] and the group SmallGroup(32,2) of order 32 [6]. The corresponding extension of G^+ is the image of a full quadrilateral group and in the Magma Library it is G = SmallGroup(32, 49) [3]. This group is generated by four involutions, labeled a, b, c and d, where r = (ab), s = (bc), t = (cd) and u = (da) and $G^+ = < r, s, t, u >$ is its orientation preserving subgroup. We will draw the regions of G.

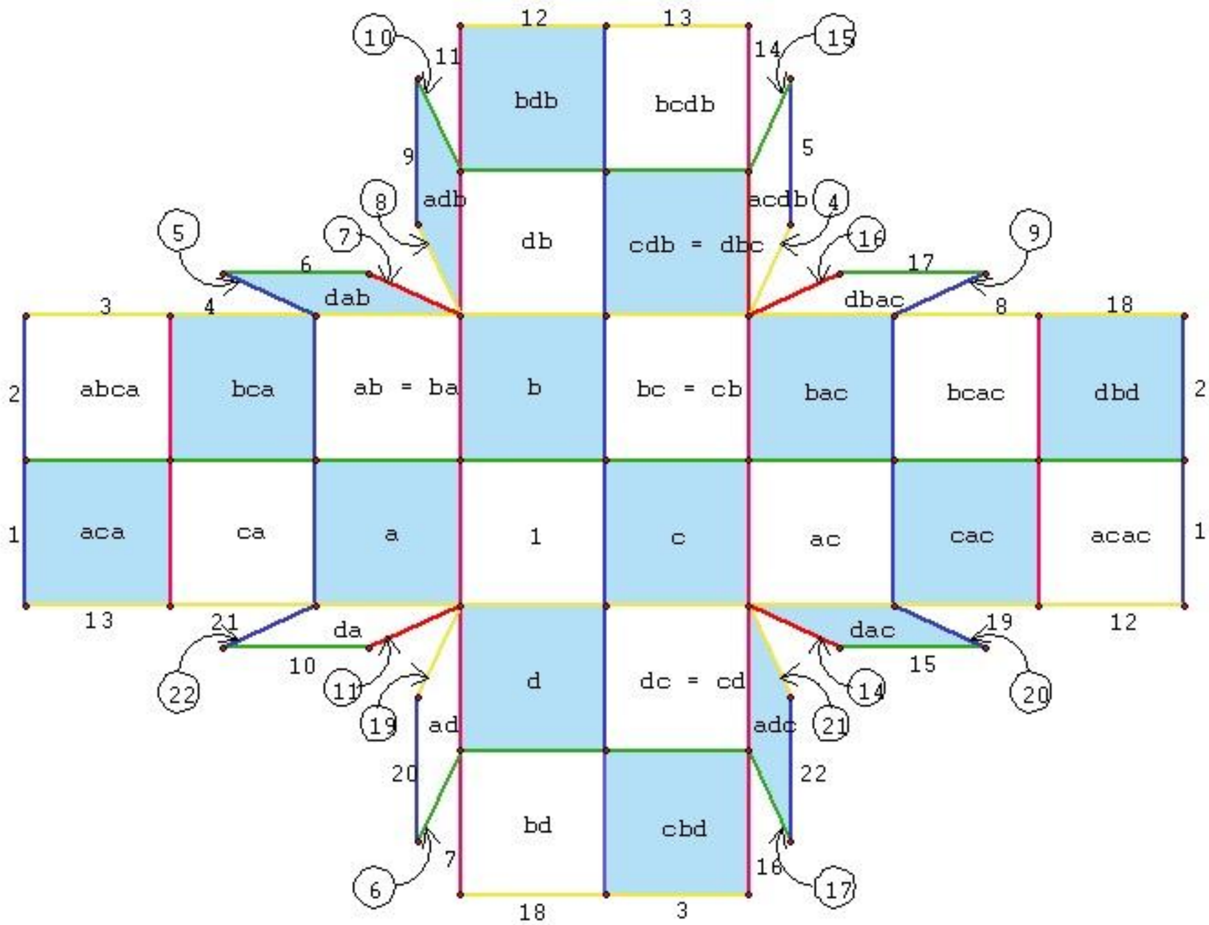

Figure 1 – Polygonal representation of G^+

In representing G in a diagram, multiplication on the right by the generator *a* is inversion in a red curve, *b* is inversion in a green curve, *c* is inversion in a blue curve and *d* is inversion in a yellow curve. Properly, all of the inversions should be reflections in "lines" in hyperbolic space and the surface is a quotient of hyperbolic space. However, the important property is the relationship between the quadrilaterals and so we will distort the quadrilaterals so that they fit into a polygonal region in the plane. This has the advantage that the quadrilaterals do not become microscopically small as they get near the boundary of the polygonal region. The polygonal region for the group G is given in Figure 1. Each edge on the rim is the same as another edge somewhere else on the rim of the diagram. The symmetric genus of G is three [3], as is the strong symmetric genus of G^+. The surface where G is drawn is divided up into 32 regions by the elements of G. Each one of these regions is bounded by 4 edges and each edge bounds 2 faces. Therefore, there are 64 edges. A vertex at the intersection of red and yellow edges has eight edges attached to it. There are four such vertices. There are 24 vertices of degree 4. They occur at the intersection of green and blue edges, green and red edges or blue and yellow edges. Red and blue edges never intersect and neither do yellow and green edges. The drawing of the group, G, has 32 faces, 64 edges and 28 vertices. Thus it has Euler characteristic -4 and hence can be drawn on a surface of genus 3.

This portrait has significant differences from previous examples. Most obviously, there are four boundary curves for each region. When drawing the portrait of a group, it helps to figure out the straight line curves. These curves are described in [6]. In this portrait there are two green straight line curves, two blue straight line curves, four red straight line curves and four yellow straight line curves. The green straight line curves are (1, 2, 3, 4, 5, 6, 19, 24) and (8, 11, 28, 17, 26, 22, 25, 15) and the blue curves are (7, 8, 9, 4, 10, 26, 18, 24) and (2, 23, 25, 16, 6, 27, 28, 14). Note that each green curve intersects each blue curve twice, alternating between the two blue curves. The same is true for the blue curves. The red straight line curves are (12, 1, 20, 5), (12, 22, 20, 11), (13, 3, 21, 19) and (13, 15, 21, 17). The red curves come in pairs, where each curve in a pair intersects twice. The same is true of the yellow curves. The yellow curves are (12, 16, 13, 14), (12, 9, 13, 18), (20, 27, 21, 23) and (20, 7, 21, 10). Notice that red curves never intersect blue curves and yellow curves never intersect green curves.

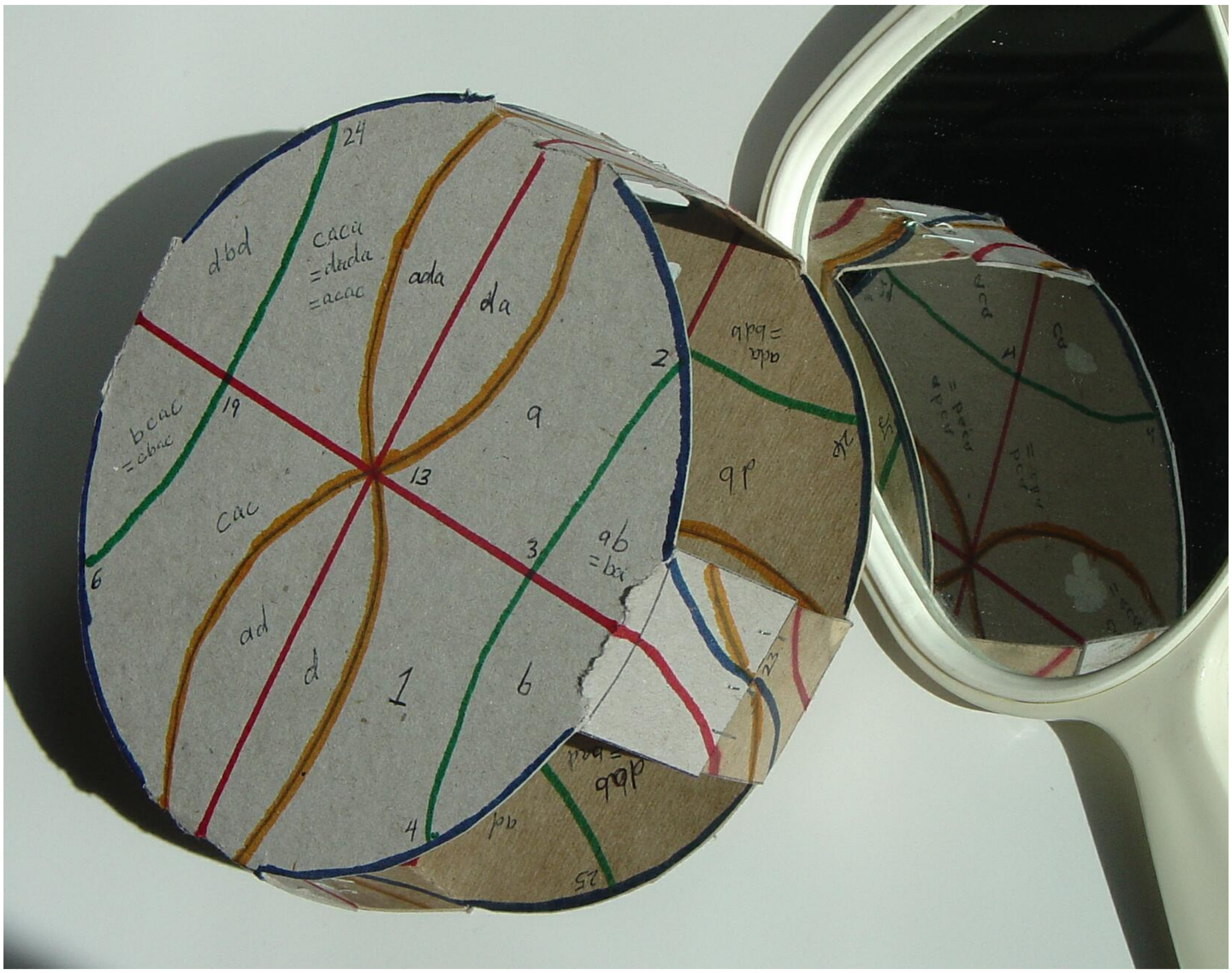

Figure 2 – Cardboard Model of G

It is easy to put the red and yellow curves in a model. If the model is like a pagoda with 4 pillars, then one pair of yellow curves go around the top and one pair around the bottom. The red curves can go down the pillars connecting the two exterior surfaces with one pair and the two interior surfaces with the other pair. This satisfies all of the information contained in just the red and yellow curves. Unfortunately, when you try to put in the green and blue curves into this model, you find that the red curves have divided the model into 4 disjoint pieces and it is impossible to put the blue curves in such a model and the same with the yellow and green curves. This problem can be fixed by drawing the yellow curves onto the pillar and down to the other level and twisting the regions around the pillar. In general, this will allow either the blue or green curves to be drawn, but not the other color. Finally, the yellow curves must be drawn onto the correct pillar and only then does the model work. The correct model is shown in Figure 2. This model has been verified by putting the correct group element into each region and verifying that all regions are adjacent to the proper region. Finally, it remains to construct a more artistic model of the compact surface.

Figure 3 – Artistic Model of G

4. References

1. W. Burnside, *Theory of groups of finite order*, (Cambridge University Press 1911).
2. H.S.M. Coxeter and W. Moser, *Generators and relations for discrete groups*, *4th Ed.*, (Springer-Verlag, 1957).
3. C. May and J. Zimmerman, The groups of symmetric genus $\sigma \leq 8$, Comm. Alg. 36, No. 11 (2008), p. 4078 – 4095.
4. J. Zimmerman, Portraits of Groups, Conference Proceedings 2006, Bridges London: Mathematical Connections in Art, Music and Science, London, England, 131 - 134.
5. J. Zimmerman, Portraits of Groups II, Orientation Reversing Actions, Conference Proceedings 2007, Bridges Donostia: Mathematical Connections in Art, Music and Science, San Sebastian, Spain, 109 - 114.
6. J. Zimmerman, A Group Portrait on a Surface of Genus Five, Conference Proceedings 2009, Bridges Banff: Mathematics, Music, Art, Architecture, Culture, Banff, Alberta, Canada, 259 - 264.