The Conformal Vega Disk

Joel C. Langer
Mathematics Department
Case Western Reserve University
joel.langer@case.edu

Abstract

The relationship between square and circle has intrigued humans since antiquity. Computer visualizations of the conformal equivalence between the two shapes double as mathematical illustrations and as *op art* after Vasarely.

Victor Vasarely created a series of *Vega* compositions, which explored possibilities for filling a disc with distorted squares, usually creating the illusion of a hemispherical bulge in a planar grid. Inspired by Vasarely (and presumably also Escher), Douglas Dunham [1] recently introduced a non-Euclidean version, in which infinitely many 'hyperbolic squares' fill the Poincare disk (the exterior of which is left blank).

Figure 1 shows another variation on the theme, based on the well-known conformal equivalence between square and disk. Though fundamentally different, the 'conformal Vega disk' shares a number of features with Vasarely's designs. The disk interior contains only finitely many 'squares' and the tiling extends naturally to the exterior (left). Visual ambiguities set off a kind of *optical vibration*: There is an underlying tension between round and rectilinear; graduated lightness towards the center hints at three-dimensionality (left); tiles appear to form vertical/horizontal rows (right) or jump to diagonal rows—albeit less dramatically than in Vasarely's *Vega* (1957) (see http://www.op-art.co.uk/victor-vasarely/).

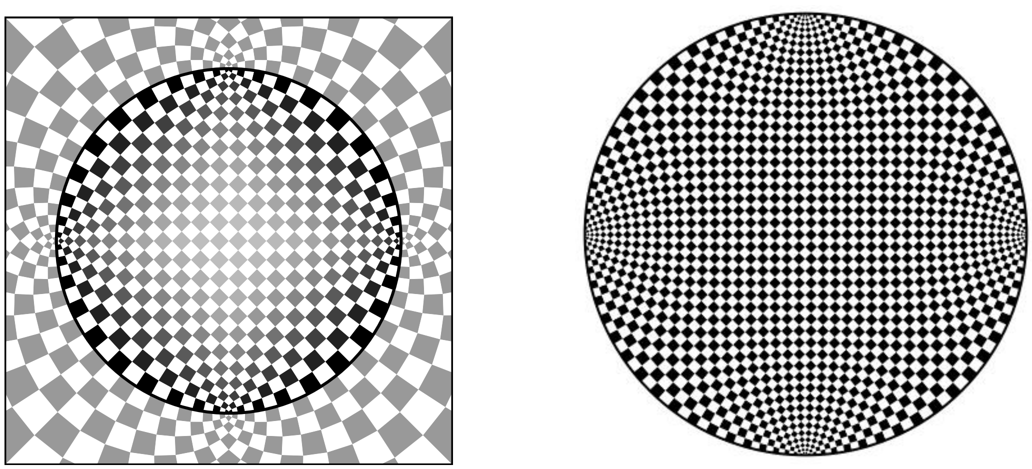

Figure 1 : *Conformal Vega disks.*

The mathematical tool for producing Figure 1 is the Jacobi elliptic sine function $f(z) = \operatorname{sn} z$ (with imaginary modulus $k = i$), in terms of which the conformal (one-to-one, angle-preserving) mapping from square to disk is easily represented. The tiling of the disk interior is the image of a tiled square. The exterior tiling (left) is the image of an adjacent tiled square, and may also be described as the 'mirror image' of the interior tiling with respect to *reflection* (or *inversion*) in the circle. A significant computational/graphical detail: For clean and efficient rendering, tiles are actually treated as quadrilaterals—the elliptic sine is used only to compute tile vertices.

Like the Vega disks of Vasarely and Dunham, the conformal version allows for diverse visual effects by choice of color scheme, tile size and variation of value (lightness); the latter two already account for the differences between left and right in Figure 1. More surprisingly, Figure 2 (top) demonstrates the visual fragility of the circle itself, which seems to get lost in the whole *conformal net* when the tiling is uniformly colored. Further, a rounded cross appears in a *cross net*, obtained via *geometric mean* from the conformal net (itself obtained via geometric mean from a pair of pencils of circles [3]). A cross, incidentally, is an old symbol for Vega, the first star ever to be photographed.

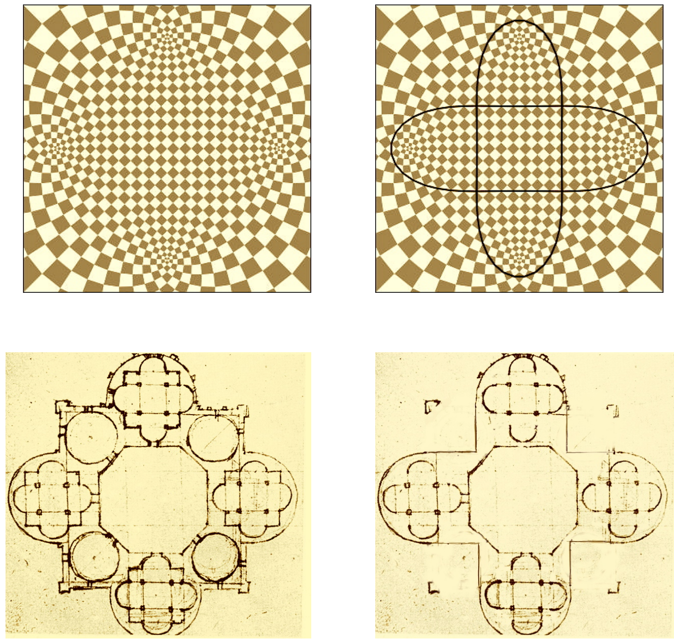

Figure 2 : *Top: With uniform coloring of tiles, circle gives way to cross.*
Bottom: Leonardo da Vinci's church plan: A study in squares, circles and crosses.

Leonardo da Vinci was, arguably, history's greatest casualty of *morbus cyclometricus*—the disease of the circle squarer. (For this diagnosis, Leonardo's lost manuscript *De Ludo Geometrico* is traditionally cited.) Among his many efforts to reconcile square and circle were artful allusions to the microcosm (*Vitruvian Man*) and to the divine—see Leonardo's plan for a church, Figure 3 (bottom left), and the same plan with digital modification, Figure 2 (bottom right). True to form, Leonardo's reach exceeded his grasp.

Indeed, our circle-square-cross figures celebrate great mathematical developments of the 19^{th} century: conformal mapping, elliptic functions, divisibility of the lemniscate [4] ($\Rightarrow$ Figures 1, 2 are constructible by ruler and compass [2]), impossibility of squaring the circle by ruler and compass, hyperbolic geometry (whose relevance here is explained in [3]). With such new ground freshly broken, it is no wonder that Escher, Vasarely and other artists of the 20^{th} century seemed to revel in a geometric world less in thrall to Euclid.

References

[1] Douglas Dunham, *Hyperbolic Vasarely Patterns*, Bridges Pécs, *88* Proceedings 2010, pp. 347–352.

[2] J. C. Langer and D. A. Singer, *The lemniscatic chessboard*, preprint (2010).

[3] —, *Checkers in the round and the lost theorem of Liouville*, preprint (2011).

[4] M. Rosen, *Abel's theorem on the lemniscate*, Amer. Math. Monthly, *88* (1981), pp. 387–395. American Mathematical Society.