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Symmetric Stick Puzzles

George W. Hart
Museum of Mathematics
New York, NY 10001 USA
E-mail: george@georgehart.com

Abstract

A large family of beautiful mechanical puzzles is based on the idea of positioning identical sticks symmetrically in space on the 2-fold axes of a polyhedral symmetry group. Many of these designs have a visual richness that gives them a sculptural quality. Structures in this family can differ in terms of the symmetry group chosen, the amount of rotation of the sticks about the axis, the length, cross sectional shape, and end treatment of the sticks, among other parameters. In certain designs, some sticks are fused together into compound parts. In addition, wherever two parts would intersect, a notch is made in one to make space for the other, and there are many options for how that choice is made throughout the puzzle.

This family of designs is explored and a simple method is presented for the reader to produce a wide variety of new puzzles. A short program written in Mathematica is used to explore designs in this parameter space and produce the overall stick geometry. Then a separate 3D editing program (e.g., Rhino) provides the constructive solid geometry operations that join the sticks, create the notches, and/or shape the stick ends. Finally, a 3D printer is used to fabricate the components. In the examples below, I use an inexpensive Makerbot to produce plastic versions of the puzzles at minimal cost. Several examples are presented of this overall process.

1. Introduction

Figure 1A shows an example of a mechanical puzzle made from twelve identically shaped sticks. By means of carefully positioned notches where they partially overlap, the sticks snap together into place. In this design, each part mates with six others and they lock each other in relative position once the final part is placed. A slight flex to the parts is required to insert the final piece. Traditionally, a puzzle designer would use trial and error to work out the notch geometry, but software presented below makes it easy to design the sticks and notches of this and many related puzzles. The user interface for the generating program (written in Mathematica) is shown in Figure 1B.

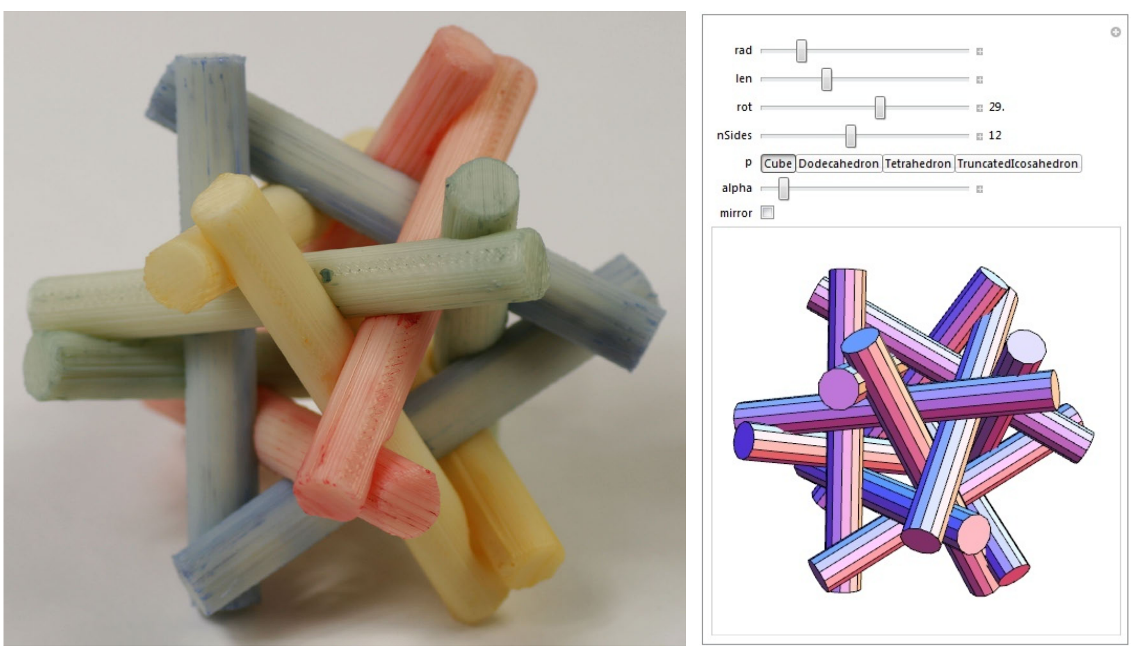

Figure 1: (A) Twelve-*stick puzzle design made of ABS plastic, 4 inches. (B) Design software.*

Figure 2. *Example wooden puzzles based on the symmetric placement of sticks. (See text for details.)*

It probably is not obvious to the casual viewer that the centers of the twelve sticks in Figure 1 are at the midpoints of the edges of an imagined cube and the angle between each stick and its corresponding cube edge is identical. As such, it belongs to a family of structures derived from a single underlying concept: start with a polyhedron and rotate each of its edges a fixed amount about the axis that connects the midpoint of the edge with the center of symmetry. By substituting shaped sticks for the rotated cube-edge segments, one has a symmetric arrangement of components which can form the basis of an attractive puzzle. There are a number of parameters which can be adjusted to give many specific puzzle designs. This paper presents the mathematical ideas and usable software for anyone to design and fabricate their own original examples in this family.

Many puzzle designers have made their own independent discoveries and creations in this broad field of puzzles. No attempt is made here to present the history of these designs, to trace the influence of various designers on each other, or to assign anyone credit for original ideas. Instead, I simply present in Figure 2 a set of twenty four examples that indicates a variety of commercially made puzzles I am familiar with in this family. The designers and fabricators include Arjeu, Stewart Coffin, Bill Cutler, Phillipe Dubois, Hiroshi Iwahara, Akio Kamei, and Tom Lensch. The photos (except Fig. 2H) are by Nick Baxter [1], who provides a rich data resource from a series of mechanical puzzle auctions. Some of these puzzles, in beautiful woods, are considered works of art by puzzle collectors and sell for thousands of dollars. But the techniques presented here allow one to make functioning plastic versions very affordably. I have not seen all of the Figure 2 designs in person, so some may have “tricks” to them beyond the geometric features described here.

Figures 2A-L are based on the twelve edges of a cube, Figures 2M-U are based on the thirty edges of a dodecahedron, and Figure 2V is based on the ninety edges of a truncated icosahedron. In 2W and X, the structure and its mirror image are combined, with two diameters of stick. Fig. 2 does not include related examples with magnets, such as the Tetraxis and Hexaxis puzzles by John and Jane Kostick. Coffin [2] is an excellent source for puzzle designs and construction techniques, but it does not deal with the range of rotations considered here. Slocum [12] describes the internal workings of some examples. Also see [8].

2. Geometry

The geometry of all these puzzles starts with the edges of a polyhedron. It is easiest to understand the case where the edges' midpoints lie on the 2-fold axes of the polyhedron's symmetry, e.g., the cube, dodecahedron, or tetrahedron. As will be seen from the derivation below, the octahedron gives the same results as the cube and the icosahedron gives the same results as the dodecahedron, so the octahedron and icosahedron are not considered separately here.

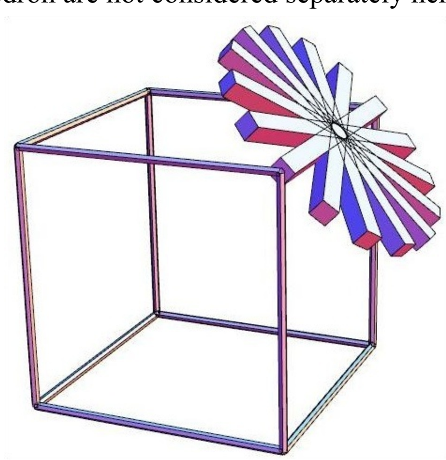

Figure 3: *Rotational positions of one cube-edge stick.*

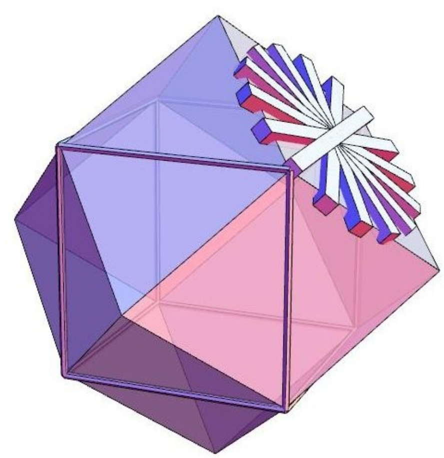

Figure 4: *Positions relative to rhombic dodecahedron.*

Consider the cube's twelve edges. Figure 3 shows the process of rotating one of the cube's edges about the line that connects its midpoint to the center of the cube. In the puzzle designs we are considering, we rotate all the edges the same amount in the same direction (as seen from outside). Because we rotate about a line from the cube's center to the edge midpoint, each rotated edge remains in one of the twelve face planes of a rhombic dodecahedron, as seen in Figure 4. (People who have made spherical tensegrity structures will be familiar with this edge rotation idea in that context; it corresponds to how far along one tensegrity strut to connect the neighboring struts [3].)

Different choices for the rotation angle give very different visual impressions, as seen in Figure 5. If we define the phase of the rotation so 0 degrees gives the cube, then a few degrees of positive or negative rotation gives a structure which is recognizably a cube with a clockwise or counterclockwise twist to the edges, e.g., Figs. 5A, B, and Fig. 2J. Rotating the edge 90 degrees gives the dual to the cube, the octahedron, in Fig. 5F. And a rotation which is a few degrees more or less than 90 gives a structure which is recognizably an octahedron with a slight twist to the edges, e.g., Fig. 2L. Between 0 and 90 lies a spectrum of intermediate structures which are not as immediately recognizable. One very special case is that at about 35.25 degrees, groups of three sticks are coplanar. Their length can be adjusted so they join to form an equilateral triangle and the result is the "orderly tangle" of four triangles as in Fig. 5C, D, and Fig. 2H [4]. The design shown in Figure 1 is six degrees from this special angle, so the four triangles can be perceived, but each consists of three edges that meet with a slight twist. Another special case occurs at about 54.75 degrees, where sets of three edges become parallel. Then each of the twelve segments is parallel to one of the four "tetrahedral directions," i.e., each is parallel to one of the long diagonals of the underlying cube, e.g., Fig. 5E and Figs. 2A-G. (The exact angles for these two special configurations are half "the tetrahedral angle" of 2 ArcTan[Sqrt[2]] (roughly 109.47 degrees) and its complement.)

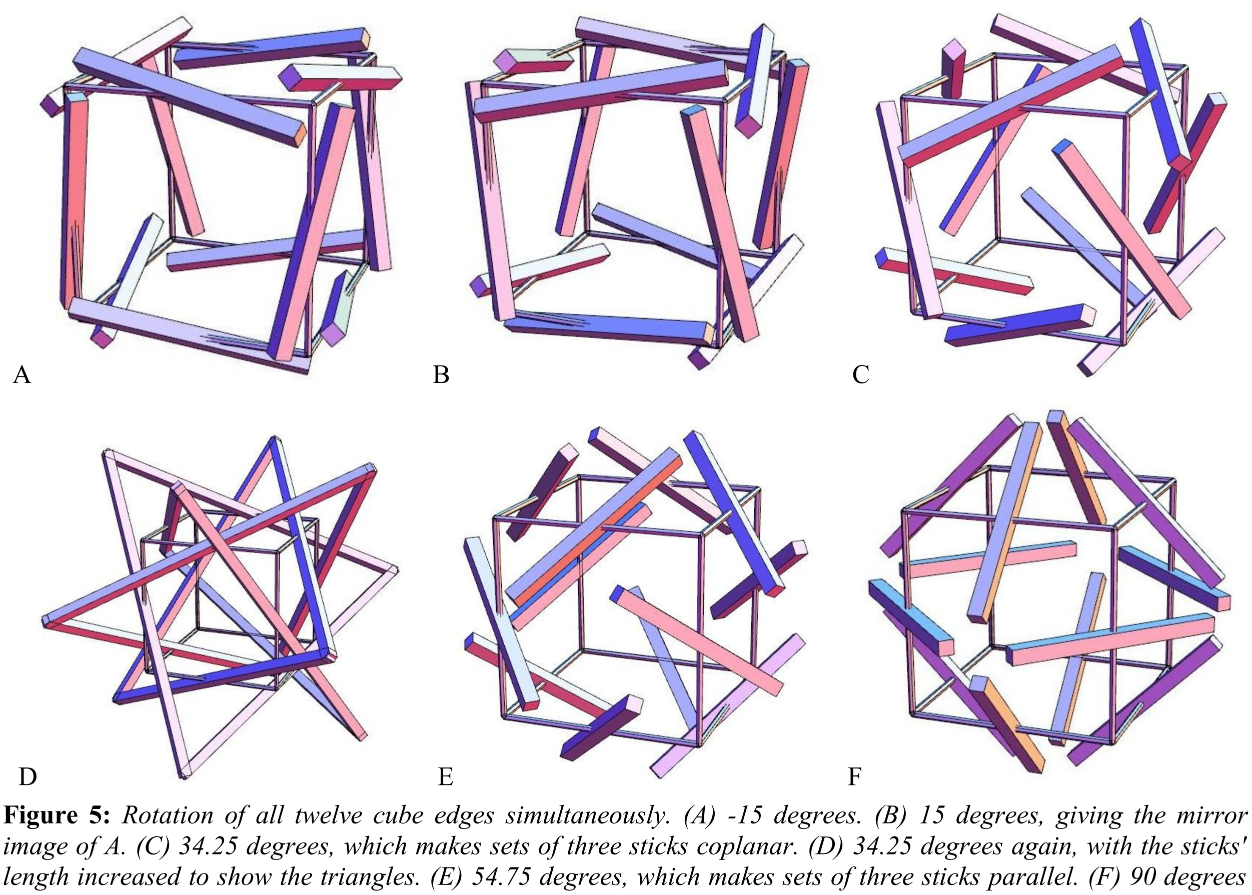

Figure 5: *Rotation of all twelve cube edges simultaneously. (A) -15 degrees. (B) 15 degrees, giving the mirror image of A. (C) 34.25 degrees, which makes sets of three sticks coplanar. (D) 34.25 degrees again, with the sticks' length increased to show the triangles. (E) 54.75 degrees, which makes sets of three sticks parallel. (F) 90 degrees giving the dual to the cube, an octahedron.*

Studying the variety of structures arising for edge rotations in just the range 0 to 90 degrees is sufficient to understand all the possibilities. This is because rotations between 0 and −90 are the mirror images of these and rotating a line segment by 180+Θ is equivalent to rotating it by Θ. The intermediate rotations, ones away from the cardinal points, are the ones I find most visually engaging and lead to puzzle designs which are the most difficult to conceptualize during assembly. They are also the most difficult for the designer to work out the geometry of the intersection of the parts, and so these are the cases where the software described here is most valuable. Starting from the dodecahedron, the edges can be rotated 90 degrees to give an icosahedron, and there are special angles making the orderly tangle of six pentagons [4] or six groups of five parallel edges, e.g., Figure 2R, S, or U.

Each possible choice of underlying polyhedron and edge rotation angle is the foundation for many puzzles, depending on how the edge is thickened into a stick. For a round stick as in Figure 1, we need to specify only its length and diameter. More generally, the cross section need not be circular, and we could theoretically choose any shape as the cross section. However, circles, regular *n*-gons, and rectangles (of various aspect ratios) are the most natural and are found in most of the commercial puzzles of Figure 2. When making such puzzles physically from wood or other stock, a constant cross section is natural for fabrication efficiency. But these techniques are also applicable to "stick-like objects" with cross sections that vary along the length of the stick, as they can readily be produced on 3D printing machinery.

After selecting the stick rotation angle, cross section, and length, another issue to consider is how to terminate the ends. A simple cross cut (orthogonal to the length) is easiest if woodworking, but all kinds of bevels may be made by mitering. Sometimes a miter for the stick ends is possible in which they join visually with other components of the assembled puzzle, e.g., Figs. 2D, F, or U. For circular sticks, hemispherically rounded ends are a natural option. When designing parts for 3D printing, there are many options for terminating the sticks that would be impractical with woodworking equipment.

If there were no contact between sticks, the puzzle could not hold together, and the designer would need to modify the parameters. Similarly, sticks in loose contact do not hold together. So, in some designs, groups of two or more touching sticks are fused into a single piece. With wood, one uses glue; with software, one uses the Constructive Solid Geometry (CSG) operation of Boolean union. Often the chosen parameters cause parts to overlap in places, as in Fig. 1. Where two sticks are visualized as overlapping, the physical material must be removed from one or the other so there are not two pieces of matter in the same place. This is an opportunity to design mating notches in various ways, e.g., there is a half-lap joint in Fig 2H, but the simplest is just to "subtract" one stick from the other. In this CSG operation, the material in the intersection of the two sticks is removed from one of them, creating a notch which meshes with the exterior of the other (unnotched) stick. Because of the symmetrical arrangement of the sticks, the intersections are also symmetrically arranged around the polyhedron center. Notches can always be made so the sticks are all identical and the puzzle has an elegant symmetry. Or as a variation, each overlapping region can be arbitrarily assigned to one of the two sticks, resulting in a puzzle with a variety of different piece shapes, which introduces additional difficulties for the solver. A puzzle with convex components can always be disassembled, but I know no automated means to determine how rigid a puzzle with notched sticks will be. One simply tries it and learns from experience.

3. Implementation

When using commercial software for 3D design, it is common to work with several packages on one project, because each has different features. For this work, I found it convenient to use Mathematica [10] for the interactive creation and visualization of the stick geometry. But Mathematica does not have built-in CSG operators, so, to make the notches, I found it easiest to export the stick design as a mesh from Mathematica and import it into Rhino [11], which has efficient stable operators for subtracting one mesh from another. Many other CAD programs might suffice for this latter operation, however CSG of meshes is difficult to implement and several other commercial programs I have tried crash at this task.

Figure 1B shows a screen shot of a Mathematica program I wrote to create the sticks. The underlying code is only two dozen lines and can be downloaded from my web site [6]. Mathematica's Manipulate operator creates a user interface with sliders that allow the real-time exploration of the design parameters [5]. It is fast and easy to change the program and explore additional parameters, so this is just a snapshot of my code development, with the following features: By clicking on one of the buttons, the underlying polyhedron is selected to be a cube, dodecahedron, tetrahedron, or truncated icosahedron. Sliders adjust the rotation angle, length, diameter, and cross section of the sticks. The number of sides can be set to three for triangular sticks, four for square sticks, etc., up to a large enough value that it appears as a circle. An “aspect ratio” slider allows the circle cross section to be changed into a tall or wide ellipse and changes the 4-sided cross section to vary from a square to a tall or wide rectangle.

As the sliders are adjusted, the visualization changes accordingly. The view can be rotated with the mouse to examine it from any side. Figure 1B shows how to replicate the design of Figure 1A. I pressed the “cube” button, which determined that there are twelve sticks. I explored with the slider and ended up with a 29 degree angle that approximates three triangles. I chose twelve sides per stick to create a roughly circular cross section. I picked the radius so there is enough overlap to lock the parts together. And I adjusted the length slider until I liked the aesthetics; each stick extends just slightly beyond its outermost intersection with another stick. Any of these parameters could easily be changed.

When the parameters are set as desired, executing an Export command in Mathematica creates a file with the stick arrangement in mesh form. Several file formats are suitable for this as long as the user chooses one which Mathematica can export and Rhino can import. I usually use the *stl* format, which provides what is needed if I also want to create the entire assembled puzzle on a 3D printer.

To join sticks, create notches, or shape the end of the sticks, I import the mesh into Rhino and use its CSG operators. There are many ways to make notches. Figure 6A shows an example in which I have replicated Stewart Coffin's “Hexasticks” puzzle of Figure 2A. One Hexastick part has been selected and a CSG operation is about to be made in which two other sticks are subtracted from it. This produces the common Hexastick piece with two notches, shown at top in Figure 6B. Depending on which other sticks were chosen to be subtracted, one, two, three, or four notches might be made in any stick, giving a variety of possible puzzle pieces, some of which are seen in Figure 6B.

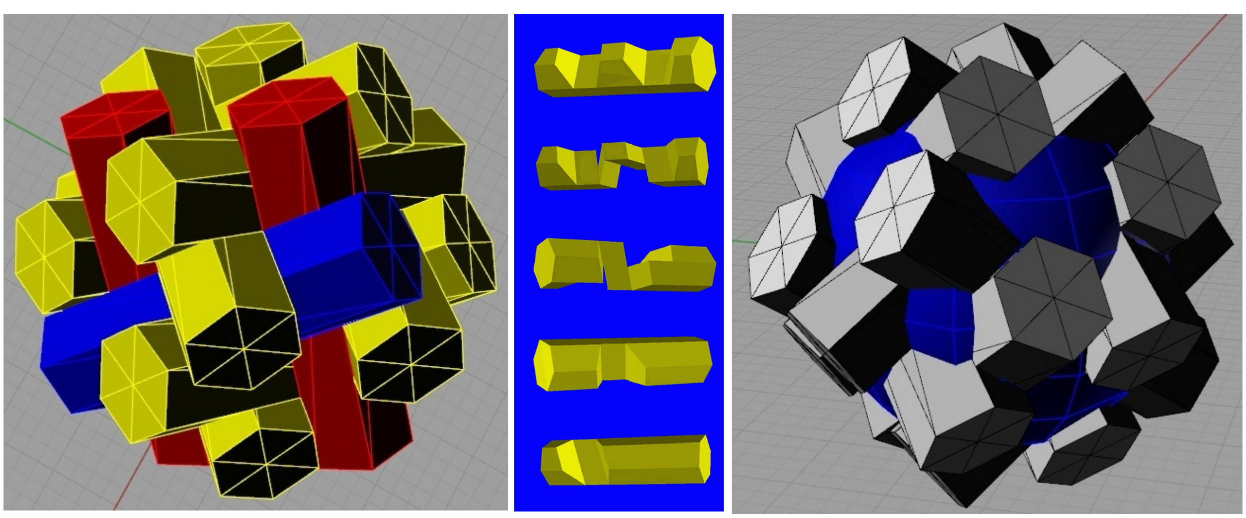

Figure 6: *(A) Selecting sticks for CSG in Rhino. (B) Assortment of possible notches. (C) Sphere to trim stick ends.*

Within a 3D design program like Rhino there are many operators available that may be used to shape the ends of the sticks. In order to make my own variant of this classic puzzle, I made some nonstandardly notched parts shown in Figure 6B and I trimmed it all with a sphere as in Figure 6C. After trimming, the sticks have a portion of a sphere as their end caps, so when assembled it is shaped like a ball. Note that this spherical end would not be simple to fabricate in wood, but is easy for a 3D printer to make.

Splitting sticks lengthwise, as in Figures 2E, F, and U, gives two half-sticks per edge. Creating these and joining groups of them into a single part is straightforward in Rhino, but not illustrated here.

The final step in Rhino is to export *stl* files of the parts for 3D fabrication. The *stl* files can be sent to a 3D printing service bureau for fabrication. Alternatively, one could fabricate wood or metal pieces from these designs, but this requires skilled craftsmanship and often one must first make special purpose jigs for accurate cutting [2]. CNC routing is also possible starting with *stl* files, but I have not tried it.

4. Examples

In the examples of this paper, I used my own Makerbot [9]. This is one of a new generation of inexpensive 3D printers aimed at the hobbyist and maker market. While it has many limitations compared to industrial machines, it cost under $1000 and is adequate for the basic production of many forms, including these puzzle parts. My model has a conveyor belt for its build platform, which lets me set up a series of parts which are built one after the other without my intervention and deposited into a bucket. Compared to more expensive commercial 3D printers, the parts are somewhat rough and some sanding and scraping is usually required to clean them up so they fit together well. So this process is not for mass production, but it is quite suitable for making one-of-a-kind puzzles and prototypes. Typically 15 to 25 minutes is required per part for the puzzle pieces in Figure 1 and below—a quick delivery time.

Figure 1 shows the first puzzle I made by this technique. I used an uncolored ABS plastic and tried to dye the parts in four colors. The dye adhered weakly, resulting in pastel colors. My plastic version of the spherical Hexastix variant of Figure 6 is shown in Figure 7. Figure 8 shows a dodecahedron-based 30-stick design with square sticks and an intermediate angle. Its outer form is similar to Figure 2R, although I have not seen that puzzle in person, so I do not know if its notches are the same as mine.

Figure 7: *Spherical variation on Hexasticks, 3 inches.*

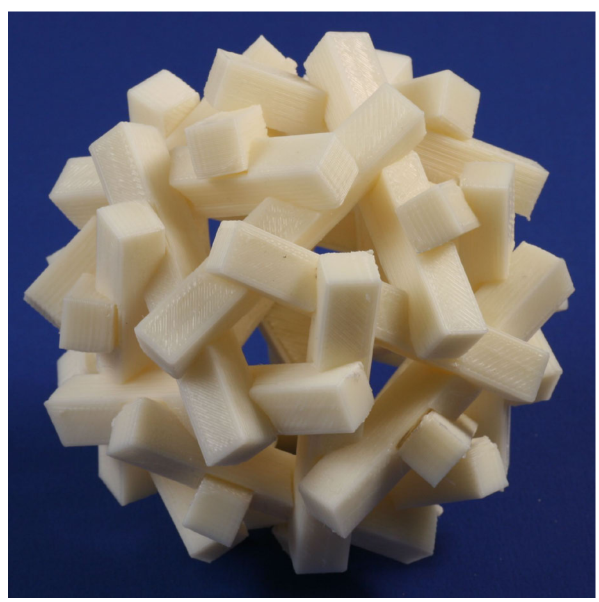

Figure 8: *30-stick puzzle of square sticks,5 inches.*

Another variation is to combine a design with its mirror image, as in Figs. 2W and X, where the mirror image structure has thinner sticks. A checkbox in the Fig. 1B software implements this feature. The mirror images can also be nested, by assigning them different overall scales. Figure 9 shows an example which combines a slightly clockwise and counterclockwise cube, e.g., Fig. 5A and B, to make a very rigid puzzle. Twenty-four sticks with the same length and diameter lie on the edges of two concentric cubes.

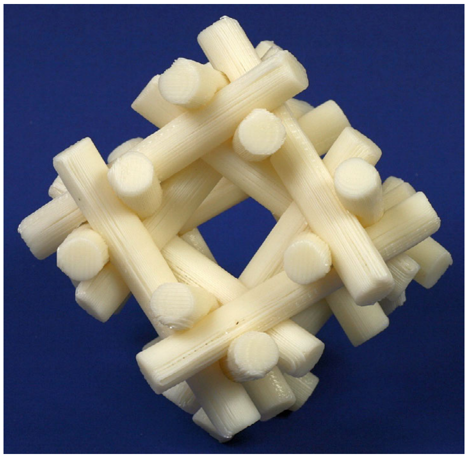

Figure 9: *Example with mirror images.*

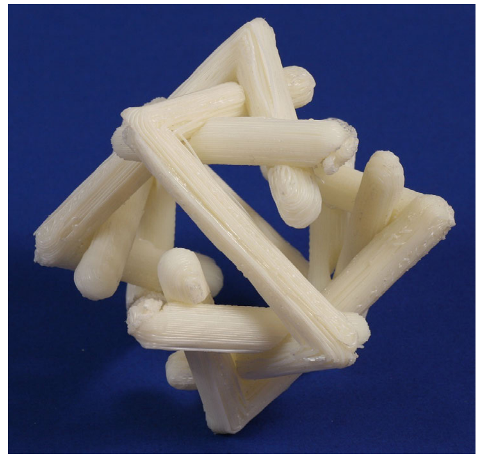

Figure 10: *Six-part truncated tetrahedron.*

If the edges of the underlying polyhedron are not all equivalent, then there will be more than one shape of piece, e.g., Fig. 2V has thirty edges of one type and sixty of another. One can take this further, based on larger Goldberg polyhedra, and work towards "Nailbanger's Nightmare" [7]. But a practical example is the truncated tetrahedron of Fig. 10. There are six pieces; each is a stick separating two hexagons unioned to two sticks separating a triangle from a hexagon, which were shortened and given rounded ends.

Many variations are possible for which there is insufficient space here, e.g., fancy notches can partition the intersection of two sticks in many ways. Simple six-piece burr puzzles are of this form. They can be derived with this software if one starts by selecting a tetrahedron as the underlying polyhedron.

5. Conclusions

With the software presented here, anyone with Mathematica can explore a wide range of symmetric stick assemblies based on rotating the edges of a polyhedron. If the designs are imported into a 3D editing program, the components are easily notched into sticks which snap together into a puzzle. Many options are possible, including joining some sticks together, or cutting sticks in half. This paper has emphasized the puzzle applications, but the assembled forms can also be considered as geometric sculpture. It is hoped that interested readers will continue exploring from here.

6. References

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