

Rigge Envelopes As Art Inspiration

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Abstract

Many artists using mathematical curves to generate lines in their work use Lissajous figures or cycloids. The mathematician William Rigge introduced a new technique in a paper in the American Mathematical Monthly in 1920 which he called envelope rosettes. In this paper, I look at what he was doing and develop some art which takes his ideas in other directions.

Introduction

The mechanical generation of curves has been a feature of many forms of art. Harmonographs, based on swinging pendulums were invented in the nineteenth century and computer versions have been used by Bob Brill in his work on Lissajous figures [1]. There were many lathe-based machines giving rise to Rose Turning machines and also gearing mechanisms (geometric chucks) which were also developed, particularly in the nineteenth century, which are essentially mechanisms of rolling circles. Robert Craig discussed one at Bridges in 2006 [3], and examples were published in a number of books in the nineteenth century, Savory [4] being a notable example. Edwards [5] gave some history and brought the subject up to date with a set of programs, in BASIC, to draw many variations. These tend to fall into the realm of more decorative and ornamental purpose, rather than on the mathematical side of the spectrum of mathematics and art. Readers will also be familiar with cycloidal designs from toys such as the Spirograph where geared circles roll on other circles; the ratio of the number of teeth means that a number of revolutions are needed to close the curve. The paper by Jack Tait of his work on programmable drawing machines which result in works he calls Taitographs [8] are at the artistic end of this spectrum, because he came from an artistic background.

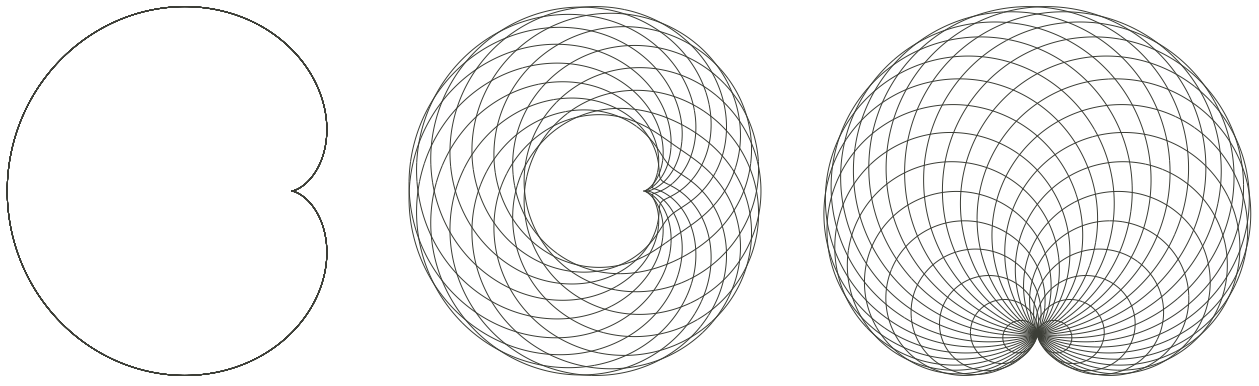
My starting point is a paper by William F. Rigge from 1920 [6] that looks at generating curves which might look at first glance to be related to the cycloids mentioned above, because he uses cycloids as his example curves. His method applies to any polar curve because he modifies the radial vector. Rigge, who was an astronomer, gave more detail and a description of a machine he built (using gears) to draw the curves in a book published by his university [7]. The Creighton machine as it is known still exists. Rigge was interested in the mathematics of the envelopes produced when he generated the curves. I am interested in the technique in order to generate artwork.

Rigge, as his book's title indicates [7] studied harmonic curves and he begins his paper [6] by saying that he is constructing curves with double harmonic motion. He starts with the polar equation of a curve called a cardioid (which can be produced as a cycloid) with polar equation $r = 1 - \cos\theta$ and modifies it thus: $r = (1 - \cos\theta) + (1 - \cos m\theta)$ where m is a constant which he keeps close to unity as a fraction, such as 15/16 or 31/32. He says that he gets a series of harmonic curves which have a common envelope. When you plot the curve, you see in fact that it gives a continuous closed curve which envelopes another curve. He also explores variations which I will denote by the following two equations:

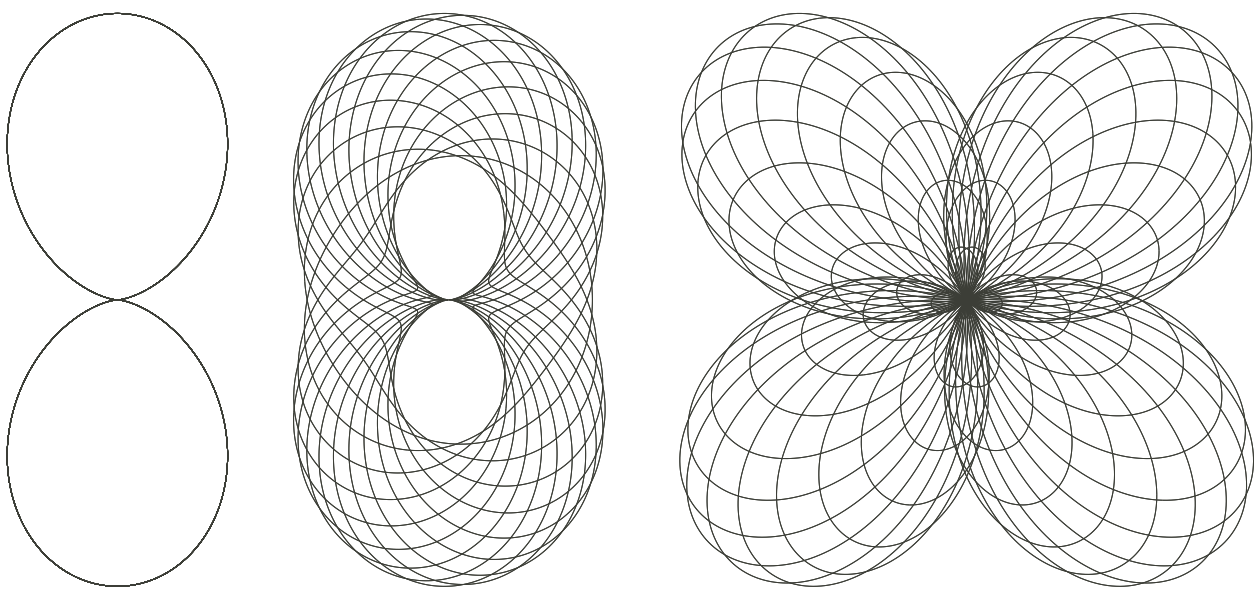
$$r = (1 - \cos n\theta) + (1 - \cos mn\theta) \quad (1)$$

$$r = (1 - \cos n\theta) - (1 - \cos mn\theta) \quad (2)$$

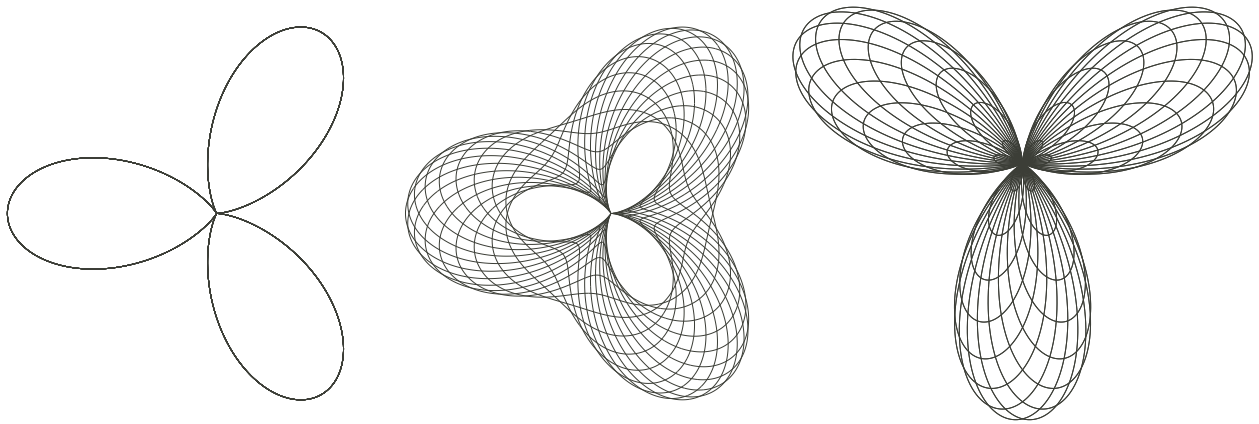
where n is an integer to give more lobes to the basic curve. Figure 1 shows some examples.



Figures 1a, 1b and 1c



Figures 1d, 1e and 1f



Figures 1g, 1h and 1i

Figure 1, The basic Rigge curves (see text for key)

The equations and variable values used in Figure 1 are as follows:

Figure 1a has equation 1 with $m = 1$ and $n = 1$, Figure 1b with equation 1 with $m = 15/16$ and $n = 1$,

Figure 1c equation 2 with $m = 15/16$ and $n = 1$,

Figures 1d, 1e and 1f correspond to 1a, 1b and 1c but with $n = 2$

Figures 1g, 1h and 1i correspond to 1a, 1b and 1c but with $n = 3$

First Variations On Rigge

An artist often produces many sketches in order to arrive at an artwork. With this type of drawing, the variables can be changed, and other polar equations used, but these tend to produce “more of the same”.

On a computer the curves are drawn as a successive series of short lines. An incremental value for θ is chosen and defines the smoothness of the line and is determined as a division of 2π , ie the equation is plotted for successive angles $2\pi/q$; in figure 1, q is 600. Making q very small alters the shape of the drawing. So in figure 2, all drawings apart from the bottom right one are based on equation 1, with $m = 31/32$ and $n = 3$. Compare the left of the top row with the bottom centre of figure 1. This shows how m changes the curve. The curve on the right in the bottom row has equation 1, with $m = 31/32$ and $n = 2$.

The value of q defining the incremental angle (see above) used in Figure 2 are as follows:

Figures 2a, 2c and 2e have a value of q equal to 100, Figures 2b, 2d and 2f have a value of q equal to 10

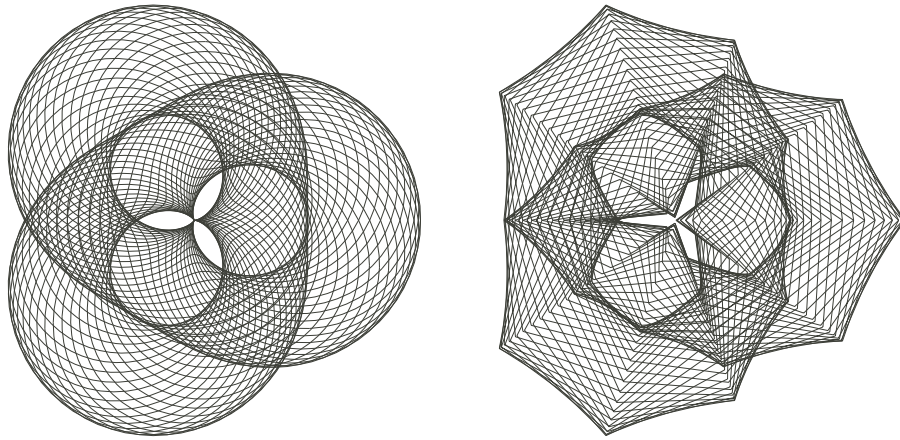
Rigge specified a value of m close to unity, using $15/16$ or $31/32$ in his examples. This changes the envelope as shown by looking at the comparable curves Figure 1h, and Figure 2a which both have $n = 2$ and respectively m being $15/16$ and $31/32$. There is no need to stick to this restriction, for example even $2/3$ gives interesting results.

Using The Circle As The Base Curve

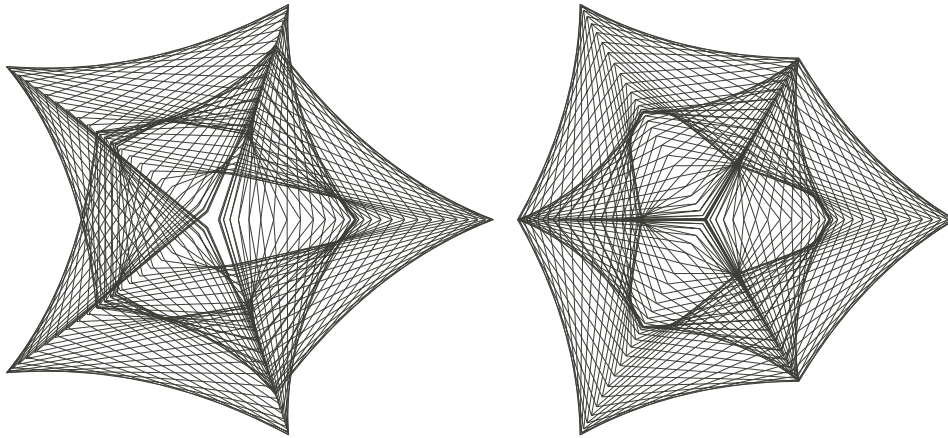
The circle might not appear to be a curve that could be employed since it has the polar equation $r = a$ with the origin at the centre of the circle. However, the polar equations $r = a\sin\theta$ and $r = a\cos\theta$ also give circles, but with the origin on the circumference; in the first case the circle is tangent to the x axis at the origin; in the second case tangent to the y axis. Using the Rigge method, both equations can be modified. Figure 3 shows $r = \sin\theta + \sin m\theta$ with $m = 15/16$. The equivalent $r = \cos\theta + \cos m\theta$ gives the same result but rotated through 90° . This is a single closed curve, but looks very similar to the standard way to draw a cardioid as an envelope by starting with a circle and a fixed point on the circle (which becomes the cusp point) and drawing discrete circles with centre on the circle which go through the fixed point. It also looks as if it is the same as Figure 1c (is it?). Varying the incremental angle, ie changing q as described above as in Figure 2, can be employed and an example is shown at the right of Figure 3.

This addition reminded me of an example in Savory [4], a version of which I had created many years ago when working on spirals. I took a circle (centred at the origin) and added a sine wave to the radius, repeating the process with a dilation and rotation to get the spiral. Figure 4 is the result, using the equation $r = a + b\sin k\theta$ with $k = 5$, $a = 1$ and $b = 0.1$ and b defining the amplitude of the wave. The two examples at the left of Figure 5, from Savory show the same effect. The first, No 49, is a correct wave; the second, No 45, appears to use the absolute value of the sine wave to make a cusp. The third example I have not analysed but is the only other example that appears close to being a Rigge curve and not a cycloidal curve.

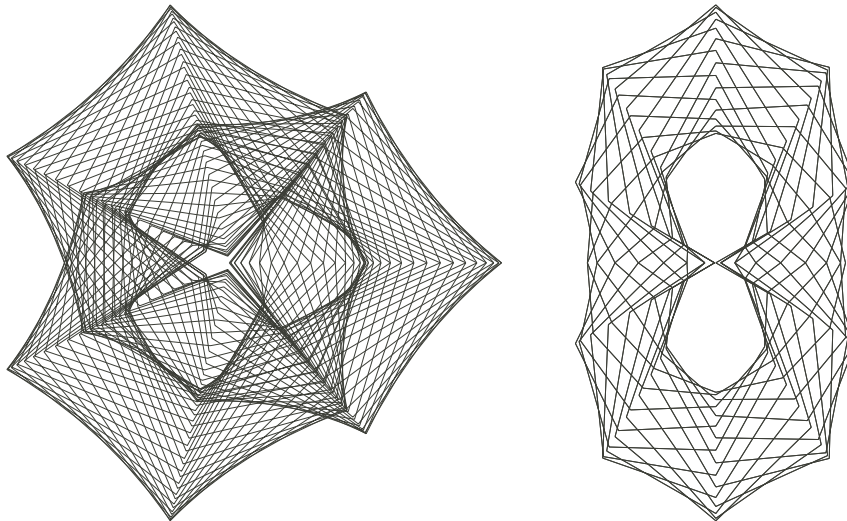
These examples suggest other variations of the Rigge method. The first would be to use the absolute value of the sine wave in one or both cases, giving three equations $r = \text{abs}(\sin\theta) + \sin m\theta$, and similarly $r = \sin\theta + \text{abs}(\sin m\theta)$ and $r = \text{abs}(\sin\theta) + \text{abs}(\sin m\theta)$. These are shown in Figure 6 with $m = 15/16$.



Figures 2a and 2b



Figures 2c and 2d



Figures 2e and 2f

Figure 2, Adding variations of incremental plotting angle, (see text for key)

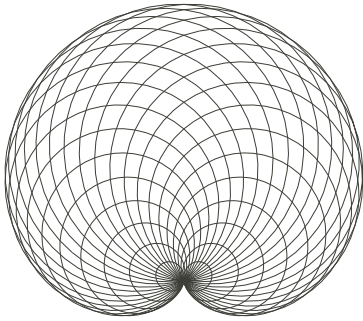


Figure 3, Rigge method applied to circle and changing increment

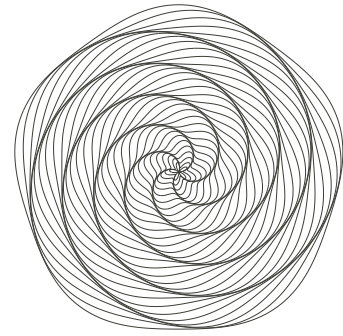
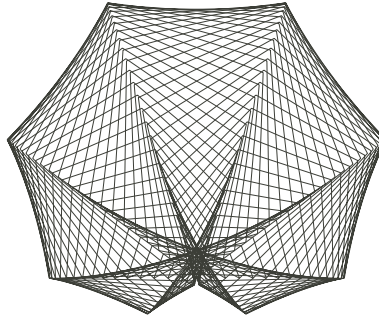


Figure 4, Wave addition spiral

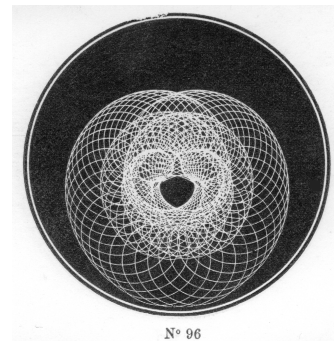
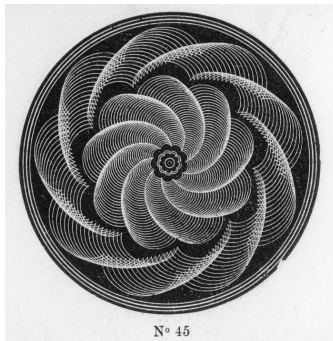
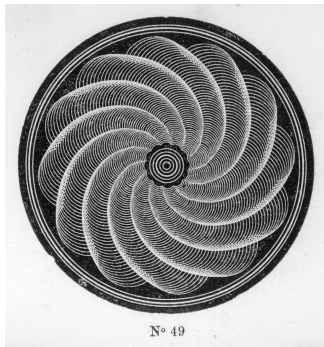


Figure 5, Examples from Savory

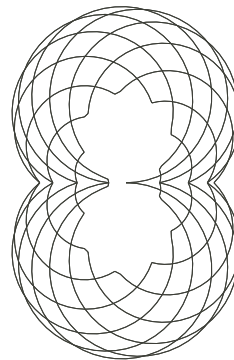
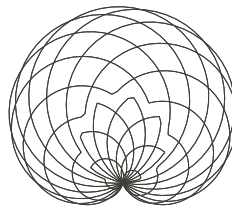
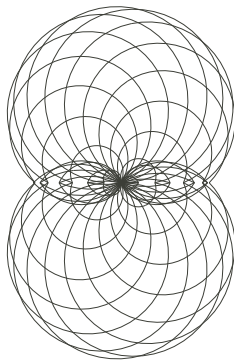
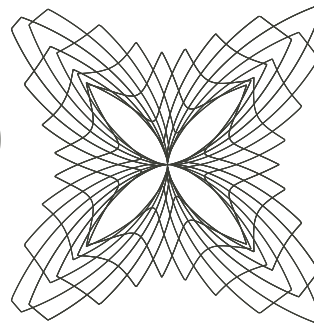
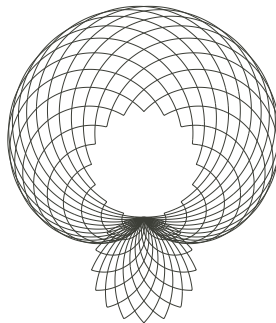
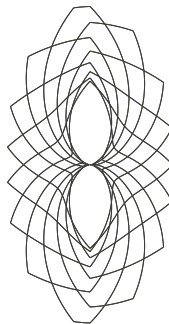


Figure 6a, 6b and 6c, Using absolute values with circles (see text for key)



Figures 7a, 7b and 7c, Using absolute values with Rigge equations

Further Possibilities In Two Dimensions

The possibilities are endless. The trigonometrical function can be made absolute as with the circle examples and Figure 7 shows some results. I created so many with small variations and have been slow in recording the parameters, so I am not sure how I produced these. Rigge's idea of getting what is similar to a precession, for example as a spinning top loses energy, by keeping m close to unity also need not be adhered to. Figure 8 shows some examples. In all cases, equation (1) is used; For Figure 8a, $m = 2/3$ and $n = 1$; for Figure 8b, $m = 3/5$, $n = 2$; Figure 8c, $m = 8/17$ and $n = 3$ and Figure 8d, $m = 17/8$ and $n = 3$. The difference in the last two is that the value of m is inverted and gives a subtle change. The second one is also surprising.

Another possibility is to mix the curves, so that additional function can be unrelated to the original one. This changes the "pattern" but not the overall shape. Figure 9 shows examples with variations on a folium, given by basic equation 4, Figure 9a. Next, Figure 9b uses equation 5 and shows the folium equation modified with a cosine term and has $m = 15/16$. Then, Figure 9c (equation 6) has a sine version of the folium equation added with $m = 15/16$. Finally, Figure 9d (equation 7) has the cardioid equation added, also with $m = 15/16$.

$$r = \cos \theta \cdot \cos 2\theta \quad (4)$$

$$r = \cos \theta \cdot \cos 2\theta + \cos m\theta \quad (5)$$

$$r = \cos \theta \cdot \sin 2\theta + \sin m\theta \cdot \sin 2m\theta \quad (6)$$

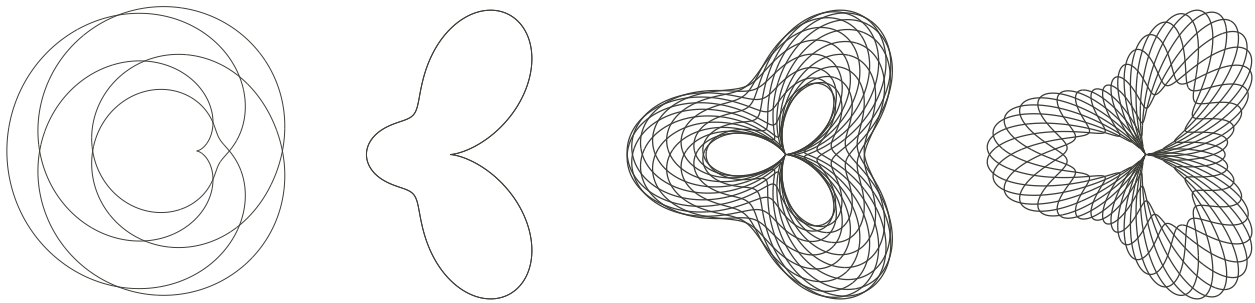
$$r = (\cos \theta \cdot \cos 2\theta) + (1 - \cos mn\theta) \quad (7)$$

Moving To Three Dimensions

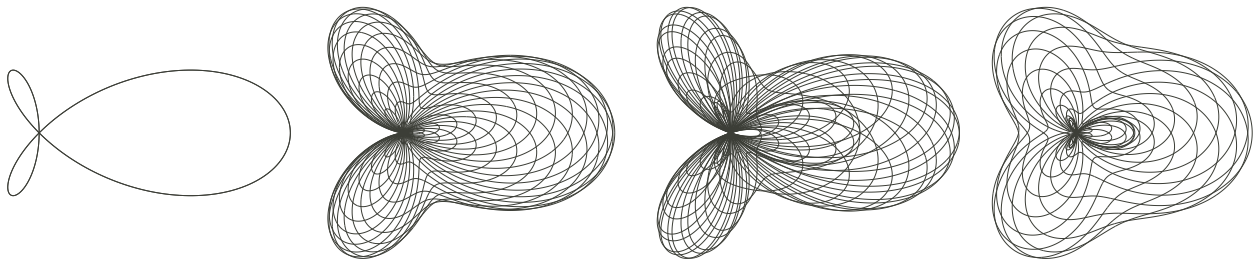
The same ideas can be applied to equations for surfaces. However, since we live in a three dimensional space, we can only see the outer hull unless the material is transparent. In the case of two dimensional curves on a plane we are looking from the third dimension, so we see the complete curve. When you are in space, one way to get some idea of the result is to cut the surface in half. An example is shown in Figure 10 which uses a spherical coordinate modification of the equation for the cardioid curve. It is rendered using the K3DSurf surface generator program [9]. This static image is not enough to understand the surface very well. You need to move it.

The second way is to apply a function to the curves to obtain the z coordinate. In plotting all the curves I used parametric equations, so that I have used a polar equation $r = f(\theta)$ and then plotted in cartesian coordinates using $x = r \cos(\theta)$ and $y = r \sin(\theta)$. For the three dimensional curve plots, I then added the third parameter as $z = \sin(m\theta)$ where m is the parameter as used by Rigge. As a thin line they are very difficult to interpret as a two dimensional snapshot and stereoscopic pairs do not do them justice either. The complexity is such that you have to move them in space so that the brain can build up the three dimensional structure. I have found VRML ideal for this. Also, by making very thick lines (placing a pipe round the lines) then rendering the image, a different effect is achieved. The cardioid curve at the centre of Figure 1 when treated this way is shown in Figure 11.

The value of m defines the density of lines. This is shown in Figure 12 where the three lobed curve at the centre of Figure 3 is created with different values of m , $3/4$, $7/8$ and $15/16$ respectively. This allows a range of sculptures to be built from a single basic curve definition. They also suggest various possibilities for jewellery.



Figures 8a, 8b, 8c and 8d, Using different values for m , (see text for key)



Figures 9a, 9b, 9c and 9d Adding different curves, (see text for key)

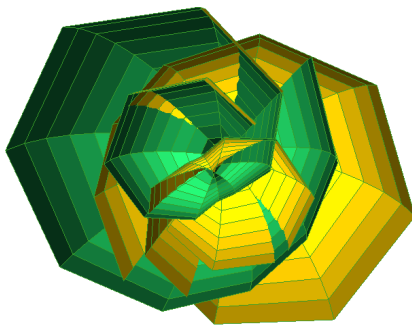


Figure 10, Cardioid based surface from K3DSurf

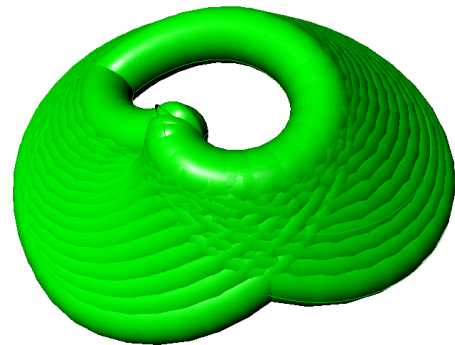
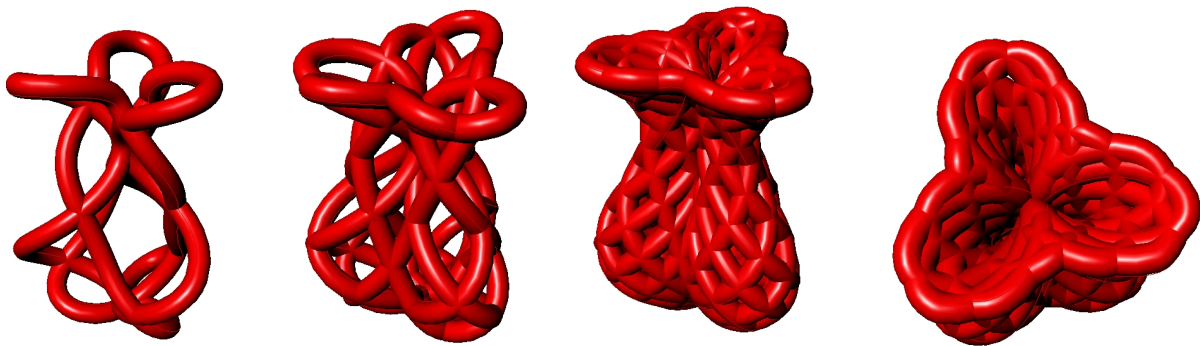


Figure 11, Cardioid surface with piped lines



Figures 12a to d, Three lobed surface with piped lines, $m = 3/4, 7/8, 15/16$, with top view

I have created a number of paper sculptures of surfaces. I usually make them from slices, hence the name Sliceforms. I looked for a way of making a Sliceform of these surfaces. The one I have made started from the Rigge envelope using an ellipse which gives the piped line surface shown in Figure 13. I created the slices by taking a series of parallel planes and intersected the three dimensional curve. The resulting points when joined gave a curve, which formed the slices. As you can see in Figure 13, the surface is open; so I joined the points on either side of the opening. The resulting Sliceform is shown in Figure 14.

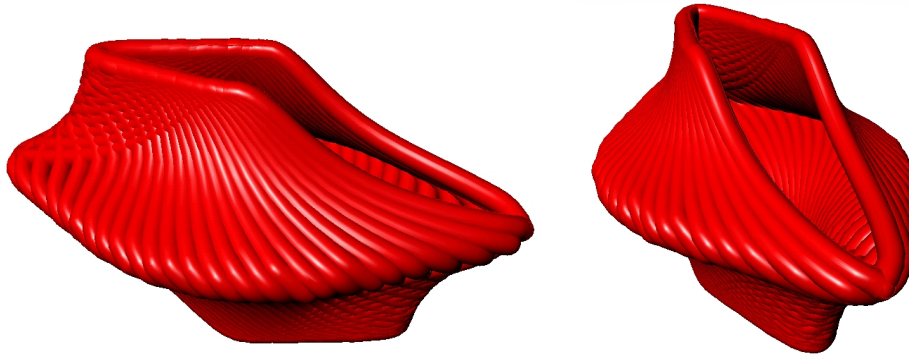


Figure 13, Surface from Rigge envelope using an ellipse, with piped lines

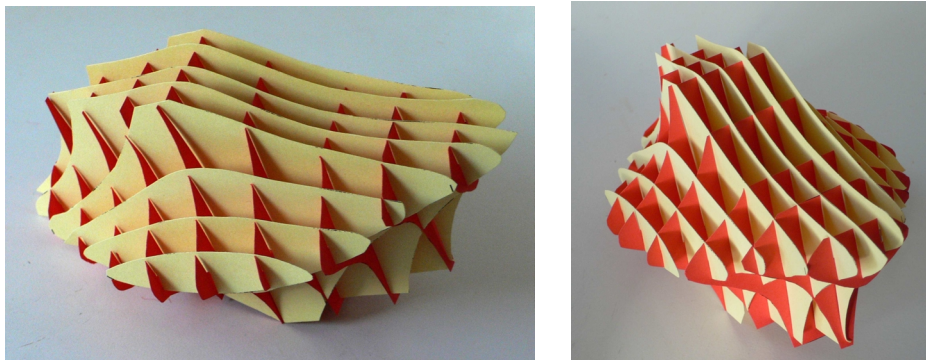


Figure 14, Sliceform version of surface in Figure 13

This paper only shows a few examples and is the tip of the iceberg on the subject.

References

- [1] Bob Brill, "The Endless Wave", Bridges Proceedings 2002, p 56.
- [2] John Sharp, "Linkages to Op-Art", Bridges Proceedings 2006, p XX.
- [3] Robert Craig, "The Mechanical Drawing of Cycloids, The Geometric Chuck"; Bridges Proceedings 2006, p 203.
- [4] H.S., *Savory Geometric Turning: Comprising a Description of the New Geometric Chuck*, Longmans, Green & Co, 1873
- [5] Ross Edwards, *Microcomputer Art*, Prentice-Hall of Australia Pty Ltd
- [6] William F. Rigge, "Envelope Rosettes" *American Mathematical Monthly*, April 1920, pp151-157
- [7] William F. Rigge, "Harmonic Curves", Creighton University, Omaha, 1926. An electronic copy can be found at www.Hathitrust.org (Accessed Jan 2011)
- [8] Jack Tait, "Taitographs - drawings made by machines", Bridges London Proceedings, 2006 p 403
- [9] The K3DSurf surface generator program is available free from <http://k3dsurf.sourceforge.net>