Specialties of Models of the 6-dimensional Cube

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Abstract

We can have several procedures to construct 3-dimensional models of the more-dimensional cubes and 2-dimensional shadows of these, even on the classical field of Platonic and Archimedean solids. The polar zonohedron models of the more-dimensional cubes can be produced either as ray-groups based on symmetrical arranged starting edges or as sequences of bar-chains joining helices. The suitable combinations of the models can result in spatial tessellations. The shadows of the models and the sections of the mosaics allow unlimited possibilities to produce planar tessellations. The moved sectional planes result in series of tiling or grid-patterns transforming into each other. Working with these methods and in search for general algorithms, we may see, even from different approaches that the 6-dimensional cube's models and their projections have more regular and more special features than those of other more-dimensional cubes and have several possibilities of application in different branches of art and design.

Polar Zonohedron Models of the *k*-cubes

We can find several procedures to construct 3-dimensional models (3-model) of the more-dimensional cubes (*k*-cubes) and 2-dimensional shadows of these. The next method of the modeling of *k*-cubes origins from a 3-dimensional reconstruction of the well known, regular octagon shaped shadow (Petrie polygon [12]) of the 4-cube [5]. Due to this result, the planar shadow of the 6-cube's 3-model can be a regular dodecagon too. Figure 1 shows the reconstructed model (with and without faces) in top and elevation views.

According to the next way to construct a 3-model of *k*-cubes, the shadow of the model can be a *k*-sided polygon in case of even *k*, but it remains a 2*k*-sided polygon if *k* is an odd number. Lifting the vertices of a *k*-sided regular polygon from their plane, perpendicularly by the same height, and joining with the center of the polygon, we get the *k* edges of the *k*-dimensional cube (*k*-cube) modeled in three-dimensional space (3-model). From these the 3-models or their polyhedral surface (Figure 2: top, elevation and axonometric views) can be generated by the well known procedure of moving the lower-dimensional elements along edges parallel with the direction of the next dimension [5, 11, 13]. Thus each polyhedron will become a polar zonohedron, more generally a zonotope [4], i.e. a „translational sum" (Minkowski-sum) of some segments [5, 13]. This structure keeps the normal cube's central symmetry and rotational symmetry too. The latter is related to the diagonal joining the starting vertex referred to the groups of any $j<k$ dimensioned elements. This diagonal is further on called as a main diagonal.

We can construct such a model of the 6-cube based on the edges of two normal cubes too. These have a common diagonal and are rotated in 60° to each other. It will be later more observable that the top view and the arrangement of the vertices in our model are similar to those of normal cubes fitted to each other by their faces.

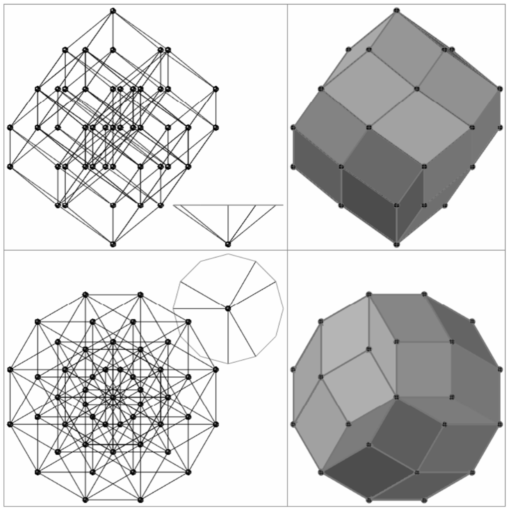

Figure 1

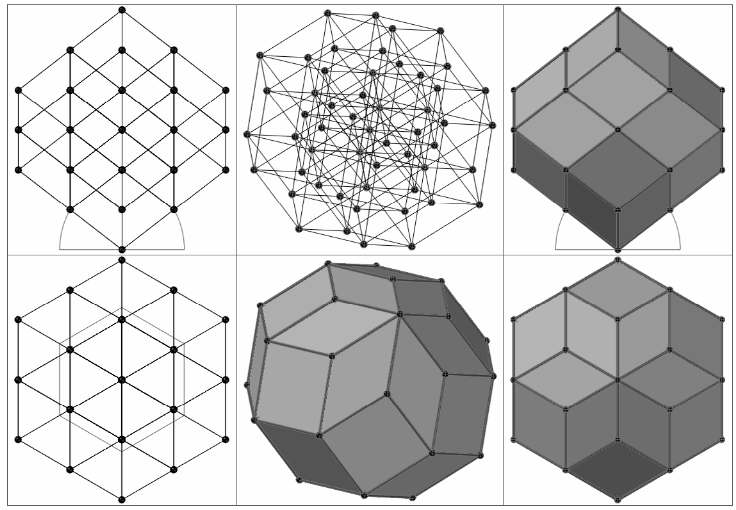

Figure 2

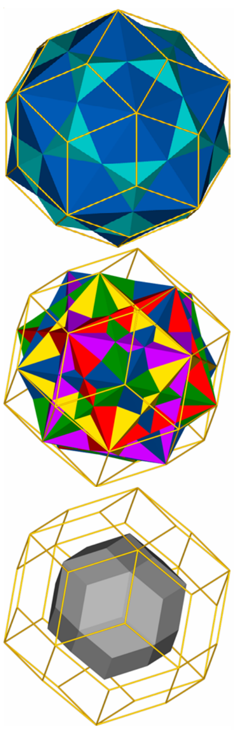

Figure 4-6

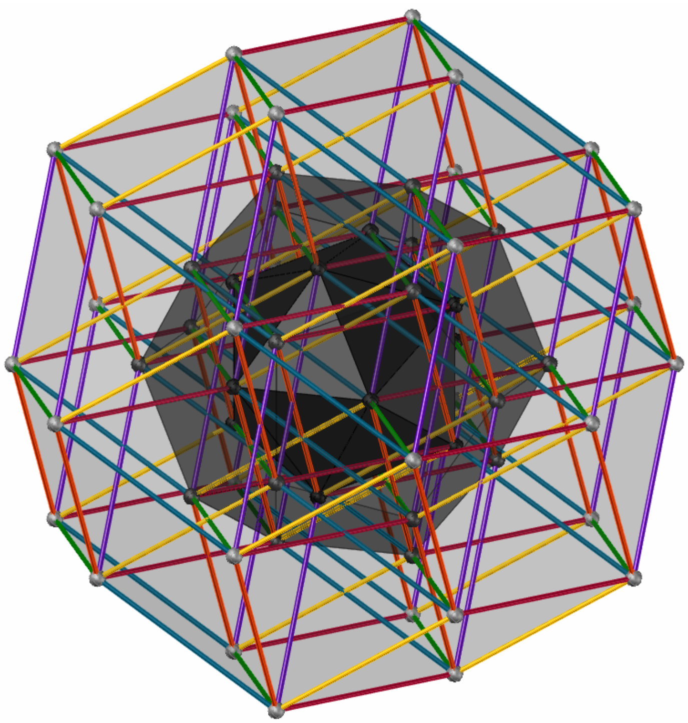

Figure 7

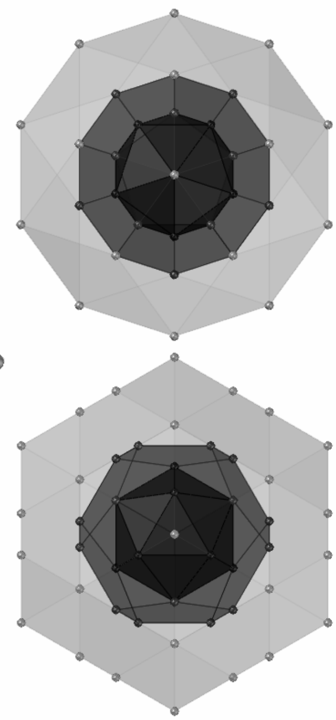

Figure 8-9

The "Golden Model" of the Six-dimensional Cube

Our first *k*-cube model, discovered the following way, was a more special 3-model of the 6-cube [5] as the above one: the common part of 5 cubes, constructed in the dodecahedron – Dh –, is a rhombic triacontahedron – RT –. If we draw parallels with every different-angled edge in each vertex of this body, we will get this model with all inner and outer edges, a 192 altogether (Fig. 5-7). The RT can be constructed by connecting the vertices of the dual pair of the dodecahedron and the icosahedron if their edges half each other (Fig. 4). The two similar RT are in proportion of the golden mean –τ– (Fig. 6). The RT-hull is defined by 32 out of the model's $2^6 = 64$ vertices. 12 out of the inner 32 vertices are joining a

Platonic icosahedron – Ii – the remaining 20 are joining a Platonic dodecahedron – Di – (Figures 7-9). The outer vertices could be arranged also on a regular icosahedron – Io – and dodecahedron – Do – (like it happened in Figure 4). The edges of these polyhedra half each other. We label the 5 congruent cubes by C severally. We can discover the golden mean concerning the edges of the next polyhedra:

$$\mathrm{Dh} / \mathrm{C} = \mathrm{Do} / \mathrm{Io} = \mathrm{Di} / \mathrm{Do} = \mathrm{Ii} / \mathrm{Do} = \tau .$$

The diagonal-pairs of the 30 congruent faces of this 3-model of the 6-cube are in proportion of the golden mean. The model is rotationally symmetric of the so called main diagonal which is perpendicular to the plane of view in Figure 8. This view has the form of a regular decagon, and holds a multitude of elements which are in proportion to each other of the golden mean. The perimeter of the model's shadow can be a regular hexagon too (Figure 9). The smaller and longer shadows of the edges are in proportion τ in this projection. We could probably find even more elements on the model and its shadows being in this proportion.

Models and Tessellations Joining Helices

Regarding the former model described in the first chapter, it is also possible to get to the endpoint of the main diagonal from the starting point along easily recognizable bar-chains, whose binding points (the outer vertices of the model) join on one helix each. The common lead of these helices is the main diagonal. The procedure to construct the whole 3-model of *k*-cubes is described in [7] taking a single helix as point of departure.

Each vertex of the 6-cube's 3-model can join on helices arranged the following way as well. Take a rotational cylinder! Shift it's copy so that it's generant coincides with the original's axis, around which 6 more polar distributed copies are created! Repeat this around all axes! Start a helix from a point of the common generants on the cylinder's surface, and mirror it to the plane defined by the generant and the axis! Distribute the helix pair around the axis six-fold! This procedure can be repeated infinitely, but the arrangement could be generated as an array of the elements too (Fig. 10 left: axonometric and top views). The vertices of the above model of the 6-cube (Fig. 2) can be joined onto the spatial distributed helices' intersection points. This very regular structure can also have architectural applications, for instance as spatial lattice girders.

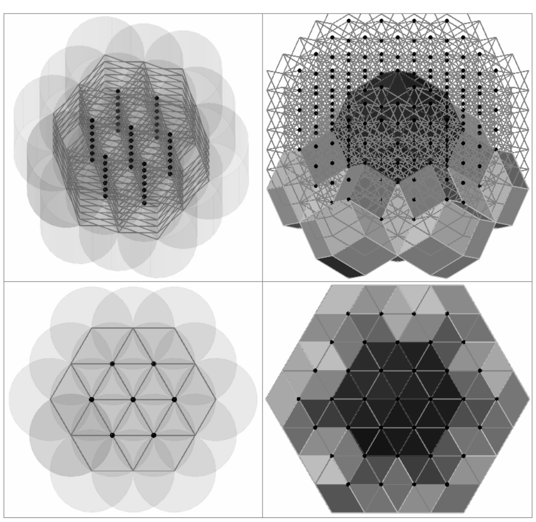

Figure 10

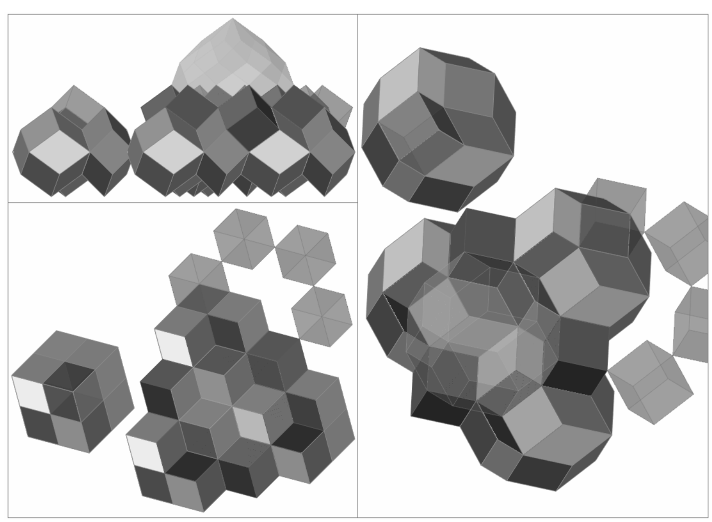

Figure 11

Combining 2<*j*<*k* edges, we can build 3-models of *j*-cubes, as parts of the *k*-cube [6]. The periodic tessellations usually investigated by us always hold the 3-model of the *k*-cube and necessary *j*-cubes originated from this. The structure of the probably densest tessellation built from the 3-model of the 6-cube and necessary derivative *j*-cubes shows Figure 11 (top, elevation, axonometric views). The *k*-models can be composed from the derivative *j*-models, thus the vertices of these join ones of the *k*-model [6]. This means that the vertices of the space-filling mosaic, based on the above described 3-model of the 6-cube or only on its *j*-cubes, join the formerly distributed helices (Fig. 10 right: axonometric and top views).

Further Tessellations Based on 3-models of the 6-cube

The above arrangement of the tessellation can be used too if we gain the 6-cube's 3-model so that we omit edges of the 3-model of a more than 6-dimensional cube. The model of the 6-cube and derivative 2<*j*<6-cubes will be constructed from the remaining 6 edges. Figure 12 shows this method for example based on the model of the 12-cube. (You can follow the figure from bottom left counter-clockwise up to the appearing of the repeated base elements.) The method is also applicable to create a space-filling mosaic based on the modified 3-model of the 5-cube originated from the 3-model of the 6-cube [9].

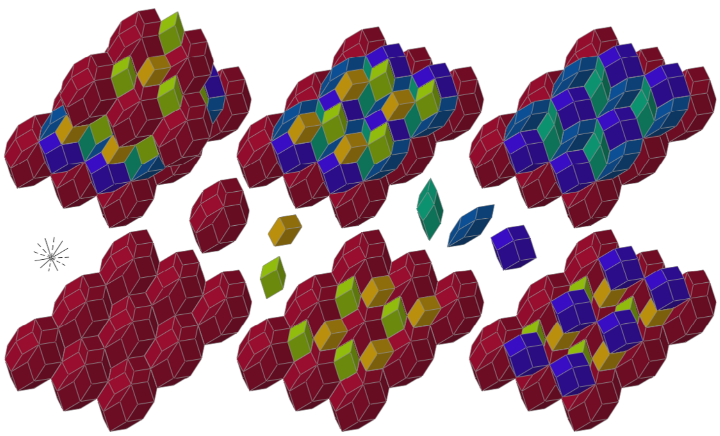

Figure 12

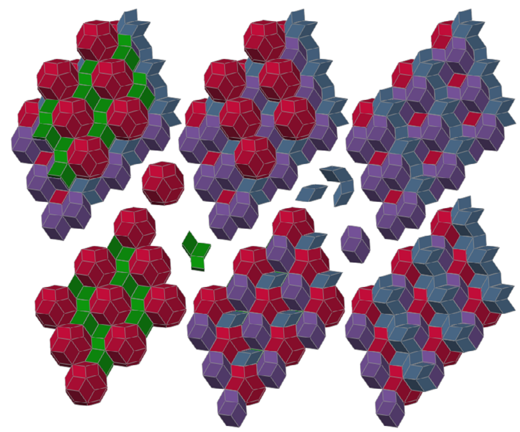

Figure 13

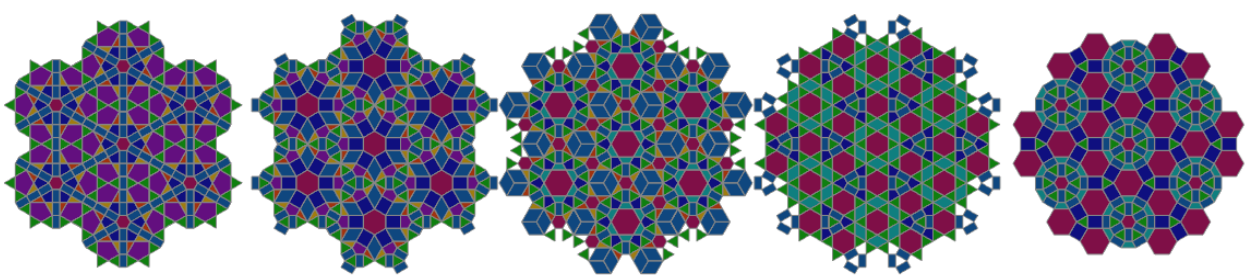

Figure 14

It can be said that we could construct a periodic tessellation which holds our regular 3-model of the 6-cube and all of its derivative *j*-cubes'. It is of course possible to create space-filling mosaics with other sets of these elements if we omit more or less from these. In Figure 14, we can see five of the 15 different main sections of a mosaic from which only one of the derivative 3-cubes' models is missing. The planes of the main sections are perpendicular to the main diagonals of the applied 6-cube models and hold the vertices of the space-filling bodies. Generally, all parallel intersecting planes hold the layers of the spatial mosaics, thus we can reconstruct the tessellations on the base of such figures if the planes are placed in the appropriate density.

Through Fig. 13 we can demonstrate as well, that a periodic tessellation can be constructed also from the RT-model of the 6-cube and its derivative *j*-cubes. (Fig. 13 can be followed like Fig. 12.) This building block set holds the elements of the 5-cube's 3-model and their pair with the additional edge parallel to the main diagonal. Two from these are constructed from the different shaped faces of the 5-cube's model.

Former proves show that by parallel sliding of edges in case of three Archimedean solids we obtain special 3-dimensional models of the 6-, 9- and 15-dimensional cubes inside these solids. This model of the 6-cube hulled by the truncated octahedron has also a special attribute because it can fill the space by itself. The structure of some spatial lattice girders is based on the former properties.

Planar Tessellations, Art and Design

The possible connections between the zonotope models of the *k*-cubes and different branches of art are analyzed more generally by the author in [7]. The 2-dimensional shadows of the models and the sections of the above described mosaics allow unlimited possibilities to produce plane-tiling which can be the base of several works of art and can help industrial designs. The moved intersecting plane(s) result(s) in series of tessellations or grid-patterns transforming into each other [8]. This can be shown through various animations as well, with the possibility of use in exhibition, publicity and so on.

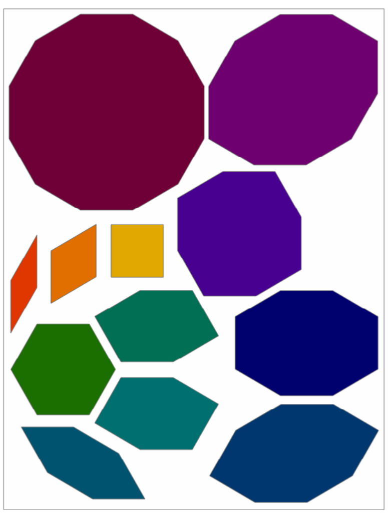

Figure 14

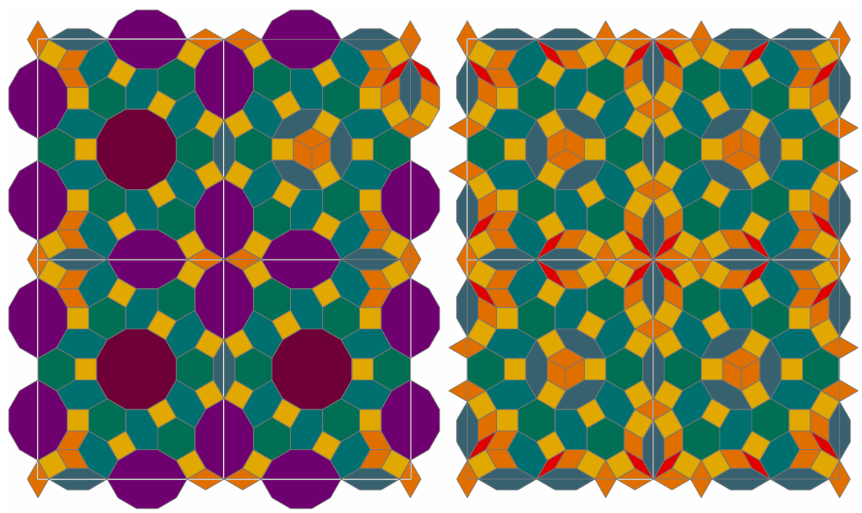

Figure 15

The parallel planar projections of the 3-models of the *k*-cubes and their *j*-cubes and faces generate sets of plane-tiling elements. We have described above the 3-models of the 6-cube which can have hexagonal and dodecagonal views respectively. The last one can generate for example the well to use tiling set showed in Figure 14. It is advisable to create a symmetric unit tiling pattern using the shadows of the derivative more-dimensional elements. We can make the constructed tessellation more complex easily if we replace the tiling elements by the shadows of their lower dimensional components (Figure 15). Thus we can grow the number of the applied planar symmetry groups.

Regarding the space-filling mosaics based on the rotationally symmetric 3-model of the 6-cube, we can compare the sections perpendicular to the main diagonal and the shadows projected parallel to this diagonal with those of the tessellation of normal cubes. These patterns or their combinations can cover each other. Several "op art" works of Victor Vasarely are based on the isometric views of the tessellation of cubes. Some of his pictures and sculptures could give the idea to create similar works of art according to the above polar zonohedron model of the 6-cube.

Remarks

This paper describes some specialties of a wide topic which naturally could not be detailed due to the necessary limit of size. Thus the most references listed below have internet accesses and can help in studying the foreground by related references as well. We try to discuss this topic among frames of the constructive geometry. It would require colored figures. You can reach more detailed text and adequate figures via http://icai.voros.pmmf.hu. The creation of the constructions and figures was aided by the AutoCAD program and Autolisp routines developed by the author.

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