Geometrical Representations of North Indian *Ṭhāṭs* and *Rāgs*

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Abstract

In his seminal works on North Indian classical music theory, V. N. Bhātkhaṇḍe (1951, 1954) classified about two hundred *rāgs* (fundamental melodic entities) by their seven-note parent modes known as *ṭhāṭs*. However, assigning *rāgs* to *ṭhāṭs* is not a straightforward task. Each *rāg* is defined by a collection of melodic features that guide a performer's improvisation. Although these features sometimes point to a unique *ṭhāṭ*, in other situations they either give incomplete information (too few notes) or give conflicting information (too many notes). Our goal in this paper is to construct geometrical models that help us to better understand the relationship between *ṭhāṭs* and *rāgs*. Following the principles of geometrical music theory (Callender, Quinn, and Tymoczko 2008), we locate the thirty-two "theoretical *ṭhāṭs*" in a five-dimensional lattice. Jairazbhoy's "Circle of *Ṭhāṭs*" connecting common *ṭhāṭs* embeds within this lattice (Jairazbhoy 1971). For a given *rāg*, our geometrical representations show which theoretical *ṭhāṭs* contain the notes used in the *rāg*'s various melodic components separately. We have written MATLAB code that produces images of a database containing a number of *rāgs*. Our models reveal graphically some of the problematic aspects of Bhātkhaṇḍe's *rāg* classification system.

1. Introduction

Rāgs are the fundamental melodic entities of North Indian Classical Music (NICM). Rather than being a fixed "tune," each *rāg* is a collection of musical features that guide a performer's improvisation. In his foundational books on North Indian music theory [1, 2], V. N. Bhātkhaṇḍe classified *rāgs* by seven-note modes known as *ṭhāṭs* (a *mode*, such as the major or minor mode in Western music, is a scale with a distinguished tonic). While there are close to two hundred *rāgs*, Bhātkhaṇḍe assigns each *rāg* to one of ten *ṭhāṭs*. Although this assignment is straightforward in some cases, quite a few *rāgs* have either too many or too few distinct notes to correspond with a unique *ṭhāṭ*.

Our goal in this paper is to construct geometrical models representing set theoretic relationships between *ṭhāṭs* and *rāgs*. While scholars have experimented for centuries with geometrical models for Western modes, including circles of major and minor modes and the Neo-Riemannian *tonnetz*, geometrical models representing the elements of NICM appeared relatively recently, chiefly in the work of Jairazbhoy [6]. Following the principles of geometrical music theory [3], we locate the thirty-two "theoretical *ṭhāṭs*" in a five-dimensional lattice. For a given *rāg*, our geometrical representations show which theoretical *ṭhāṭs* are supersets of notes used in the *rāg*'s *āroh*, *avroh*, and *pakaṛ* separately. These reflect the degree to which a *rāg* is unambiguously identified with its *ṭhāṭ*. We have written MATLAB code that produces images of a database of *rāgs*.

The basics of North Indian music theory are as follows. As in Western theory, seven notes, Sa, Re, Ga, Ma, Pa, Dha, and Ni span an octave; this sequence of notes repeats in higher and lower octaves. Of these notes, Re, Ga, Ma, Dha, and Ni have two positions, *śuddha* (natural) and *vikrit* (altered), which may either be *komal* (flat) or *tīvra* (sharp). The only note among these to have a *tīvra* position is Ma, while the rest have *komal* and *śuddha* versions. Thus the twelve notes in an octave, successively a semitone apart, are: Sa, Re (*komal*), Re (*śuddha*), Ga (*komal*), Ga (*śuddha*), Ma (*śuddha*), Ma (*tīvra*), Pa, Dha (*komal*), Dha (*śuddha*), Ni (*komal*), Ni (*śuddha*). We will use the abbreviated list {S, r, R, g, G, m, M, P, d, D, n, N} when convenient. We note that Indian note names indicate relative, rather than absolute, pitch; the performer is free to choose the actual pitch identified as "Sa."

A *ṭhāṭ* is an ordered collection of the seven notes, where only one version of each note may be selected. Since five of the notes have two positions, it is theoretically possible to create thirty-two (2^5) *ṭhāṭs*. However, only the ten *ṭhāṭs* listed in Table 1 are commonly used in NICM. Six of these, including the major (Ionian) and minor (Aeolian) modes, are known in the West as Glarean modes.[2]

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[2] Glarean modes, named for the sixteenth century music theorist Heinrich Glarean, all belong to the same *set class*, meaning that, modulo cyclic permutation or reversal, they have the same sequence of intervals between adjacent notes. This set class, the diatonic scale, has quite a few desirable properties, including the fact that it is nearer than any other seven-note collection in twelve-tone equal temperament to the even division of an octave into seven parts. In addition, it is "generated" by a sequence of six perfect fifths modulo the octave (see [4]).

Kalyāṇ	SRGMPDN	CDEF♯GAB	Lydian	*Bhairavī*	SrgmPdn	CD♭E♭FGA♭B♭	Phrygian
Bil̄aval	SRGmPDN	CDEFGAB	Ionian	*Toṛī*	SrgMPdN	CD♭E♭F♯GA♭B	
Khamāj	SRGmPDn	CDEFGAB♭	Mixolydian	*Bhairav*	SrGmPdN	CD♭EFGA♭B	
Kāfī	SRgmPDn	CDE♭FGAB♭	Dorian	*Pūrvī*	SrGMPdN	CD♭EF♯GA♭B	
Āsāvarī	SRgmPdn	CDE♭FGA♭B♭	Aeolian	*Mārvā*	SrGMPDN	CD♭EF♯GAB	

Table 1: *The ten common ṭhāṭs of North Indian Classical Music and their Western equivalents.*

rāg	*ṭhāṭ*	*āroh*	*avroh*	*pakaṛ*	*v., s.*
Āsāvarī	*Āsāvarī* = SRgmPdnṠ	**S**Rm**P**dṠ	Ṡn**dP**m**gR**S	**RmP**ndP	d, g
Mālkauns	*Bhairavī* = SrgmPdnṠ	ṇ**S**g**md**nṠ	Ṡn**dm**gmgS	m**g**mdn**dmg**S	m, S
Shuddhakalyān	*Kalyāṇ* = SRGMPDNṠ	**S**R**G**PDṠ	ṠND**P**M**G**RS	**G**R**S**ṆḌ**P̣S**G**R**P**R**S	G, D
Bhupāli	*Kalyāṇ* = SRGMPDNṠ	SRG**PD**Ṡ	**Ṡ**D**PG**RS	**GR**SḌSR**G**P**G**DP**G**RS	G, D
Kedar	*Kalyāṇ* = SRGMPDNṠ	S**m**m**P**D**P**NDṠ	**Ṡ**N**DP**MPD**Pm**GmRS	S**mmP**DP**m**P**m**RS	m, S

Table 2: *Five rāgs. The column marked "v., s." indicates the rāg's emphasized notes (vādī and samvādī). Bold letters within a melodic element indicate prolonged notes; a dot above or below a note indicates transposition up or down an octave, respectively.*

While there are only ten *ṭhāṭs* in common use, there are about two hundred *rāgs*. A *rāg* is a melodic theme upon which a performer improvises while staying within the allowable boundaries of note patterns and combinations specific to that *rāg*. Each *rāg* is characterized by its ascending and descending sequences (*āroh* and *avroh*), its "catch phrase" (*pakaṛ*), its emphasized notes (*vādī* and *samvādī*), the number of notes it contains (*jātī*), the octave emphasized, and the time of day it is performed. *Rāgs* may be pentatonic, hexatonic, or heptatonic depending on the number of distinct notes they use.

Table 2 summarizes five *rāgs* from three *ṭhāṭs*. In theory, a *rāg* is assigned to a parent *ṭhāṭ* largely on the basis of agreement of notes in the *rāg* with those of the *ṭhāṭ*. This is clearly true in the case of *rāg Āsāvarī*: the union of the set of notes in its *āroh*, *avroh*, and *pakaṛ* corresponds exactly with *Āsāvarī ṭhāṭ*. Although both its *āroh* and *pakaṛ* are "incomplete" in that they contain less than seven distinct notes, *Āsāvarī* is the only one of the ten common *ṭhāṭs* that have the *rāg*'s *āroh* or *pakaṛ* as subsets. (For example, its *āroh* contains the notes S, R, m, P, and d. Four of the thirty-two theoretical *ṭhāṭs* also contain these notes; of them, only *Āsāvarī* is a common *ṭhāṭ*.)

Identifying *rāgs* with *ṭhāṭs* based on subset relationships is not always straightforward. In particular, the notes of hexatonic and pentatonic *rāgs* are subsets of more than one *ṭhāṭ*. Bhātkhaṇḍe mentions these difficulties in his major work, the *Kramik Pustak Mālikā* [2], where he provides brief descriptions for each of about 180 *rāgs*. For example, *rāg Mālkauns* is a pentatonic *rāg* containing only the notes $\{$S, g, m, d, n$\}$. On the basis of notes alone, it could equally well belong to *Āsāvarī* or *Bhairavī*. Bhātkhaṇḍe notes that *rāg Mālkauns* "is generated from *Bhairavī ṭhāṭ* ... some say that it is in *Āsāvarī ṭhāṭ*" [2, vol. 3, p. 701, translated from Hindi].

The comparison of *rāgs Shuddhakalyān* and *Bhupāli* reveals another challenge for the practitioner of NICM. Bhātkhaṇḍe singles out certain *rāgs* that are "close" and explains what a performer must do to avoid crossing over to a neighboring *rāg*. For example, he describes *Shuddhakalyān* as similar to *Bhupāli*, but, "unlike *Bhupāli*, in this *rāg* the lower octave is used more ... In *avroh* [the note] Ni is used many times and this distinguishes it from *Bhupāli*" [2, vol. 4, pp. 60-61]. Note that *Shuddhakalyān*'s heptatonic *avroh* not only distinguishes it from *Bhupāli* but also identifies the *ṭhāṭ*. (In general, we note that a *rāg*'s *avroh* is more likely than its *āroh* or *pakaṛ* to signal its *ṭhāṭ*.)

In contrast, *rāg Kedar* has "too many" distinct notes (eight) rather than too few. It belongs to *Kalyāṇ ṭhāṭ*, even though subset analysis seems to suggest *Bil̄aval* (in particular, its *āroh* belongs to *Bil̄aval*, its *avroh* contains both *Bil̄aval* and *Kalyāṇ*, and its *pakaṛ* belongs to *Bil̄aval*, *Khamāj*, or *Kāfī*). Moreover, *Kedar*'s *vādi* (emphasized note) is a natural Ma, while *Kalyāṇ ṭhāṭ* has a sharp Ma. Bhātkhaṇḍe comments that both sharp and natural forms of Ma are used. Ancient writers did not allow use of sharp Ma in *Kedar* and considered it to be under *ṭhāṭ Bil̄aval*. Presumably, the sharp Ma "trumps" the natural.

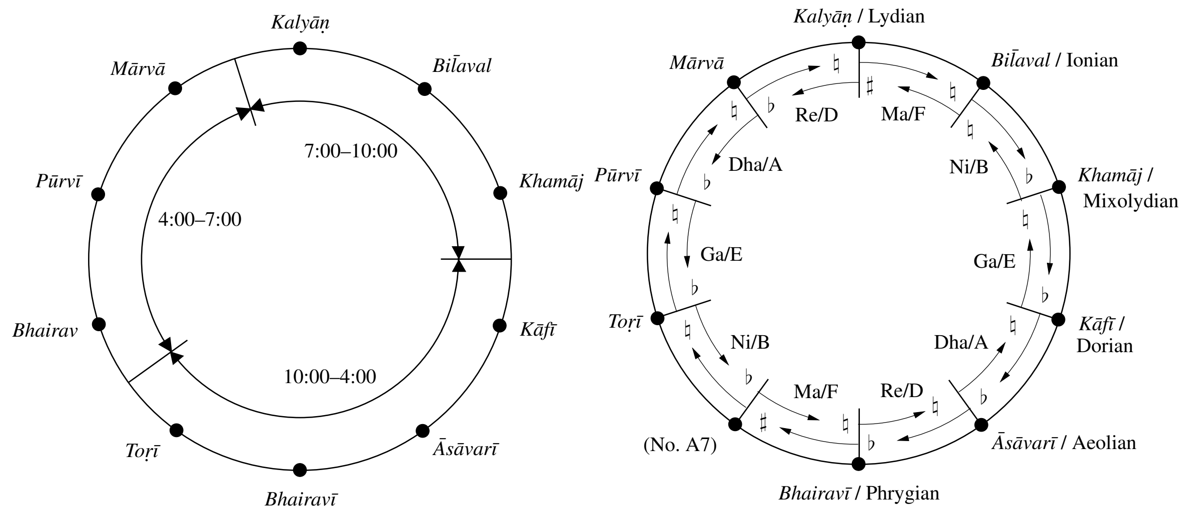

Figure 1: *Circle of Ṭhāṭs after Bhātkhaṇḍe (left) and Jairazbhoy (right).*

2. Geometrical Models

Due to the association between *rāgs* and times of day, the depiction of *rāgs* on a circle is natural. *Rāgs* belonging to the same *ṭhāṭ* are typically performed either at the same time or separated by half a day. On this basis, Bhātkhaṇḍe proposed to identify *ṭhāṭs* with times of day on a twelve-hour cycle. Jairazbhoy [6, p. 63] took the logical next step by arranging *ṭhāṭs* on a circle according to Bhātkhaṇḍe's time theory, as in Figure 1 (left). Remarkably, nine of the ten *ṭhāṭs*, starting with *Bhairav* and proceeding clockwise to *Bhairavī*, form a sequence in which each *ṭhāṭ* is related to its neighbors by a one-semitone alteration in one of its notes. For example, we move from *Kalyāṇ* to *Bilāval* by changing *Kalyāṇ*'s sharp Ma to a natural Ma (F→F♯), while we move from *Kalyāṇ* back to *Mārvā* by flatting *Kalyāṇ*'s natural Re (D→D♭). In other words, with the exception of *Toṛī*, *ṭhāṭs* that are adjacent in time are linked by voice leadings (bijections between collections of notes) which are "efficient" in that only a small amount of chromatic alteration takes place. Roy [7, p. 82] theorizes that the agreement between the ordering of *ṭhāṭs* based on efficient voice leading and the ordering based on time theory is probably due to the "tendency of *rāgas* to follow the line of least resistance in the easy transition from scale to scale . . . observed to a certain extent by all musicians." Since moving from one *ṭhāṭ* to another requires retuning some musical instruments, it is advantageous to arrange the cycle so any two neighboring *ṭhāṭs* share as many common tones as possible. In the sequence of six *ṭhāṭs* from *Kalyāṇ* to *Bhairavī*, one has the added advantage that the new pitch is always a perfect fifth from one of the notes in the original scale. (After the octave, the perfect fifth is the easiest interval to tune.) We also note that typical models of Western modes share the feature that the modes are linked by efficient voice leading [3].

Is there a way the voice leading approach can be made to include *Toṛī*? And what of the thirty-two theoretical *ṭhāṭs*: can they be incorporated into a model? We locate theoretical *ṭhāṭs* as vertices of a graph in Figure 2 (Jairazbhoy depicts a isomorphic graph in [6, p. 184]). Two *ṭhāṭs* are connected by an edge if and only if they differ by one semitone. Note that, although the graph is a convenient model for local connections between *ṭhāṭs*, it does not represent distances—each edge in the graph represents a one-semitone alteration, but the edges are different lengths. Moreover, it does not represent all possible pathways between *ṭhāṭs*.

Bhātkhaṇḍe's ten common *ṭhāṭs*, indicated by ringed circles, define a connected subgraph of the lattice. In order to complete a cycle, Jairazbhoy adds a theoretical *ṭhāṭ* labelled "A7" (so called because of his classification scheme). This move successfully incorporates *Toṛī* but leaves out *Bhairav*. Jairazbhoy's "Circle of *Ṭhāṭs*," as in Figure 1 (right), embeds as a cycle in the graph of theoretical *ṭhāṭs*. The graph also reveals the problem: *Pūrvī*, *Bhairav*, *Toṛī*, and *Bhairavī* lie on the vertices of a cube in the lattice, and there is no path that connects them all, using transitions where some note is altered by a single semitone. An alternate to Jairazbhoy's solution is to allow the path to bifurcate, connecting *Pūrvī* to both *Toṛī* and *Bhairav*, then connecting *Toṛī*, *Bhairav*, and *Bhairavī* to the unique theoretical *ṭhāṭ* that is within a one-semitone alteration of all of them. (Jairazbhoy [6, p. 97-99] cites historical and theoretical reasons for preferring "A7" to this *ṭhāṭ*, however.)

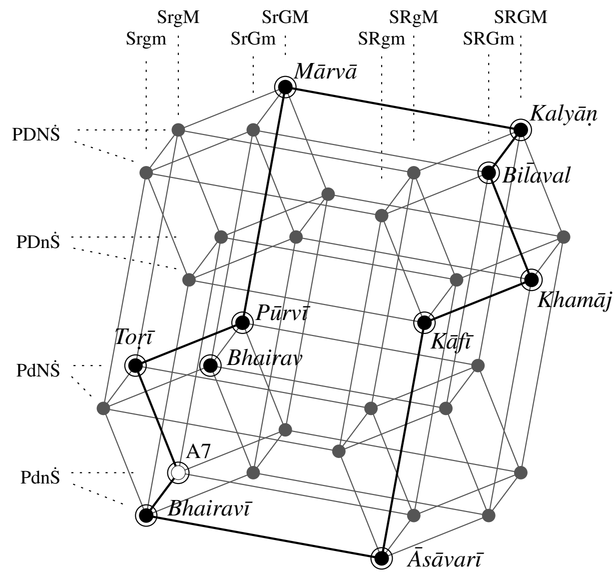

Figure 2: *Lattice of thirty-two theoretical ṭhāṭs.*

Geometrical music theory provides a way of thinking about geometrical representation in general (see [3]): any musical object that can be represented by an n-tuple of pitches corresponds to a point in some n-dimensional Euclidean space. Equivalence relations, such as octave equivalence, define quotient maps on Euclidean space producing a family of singular, non-Euclidean, quotient spaces—*orbifolds*. Points in these spaces represent equivalence classes of collections of notes, such as chords or scales. Any voice leading corresponds to a line segment or path in an orbifold. In order to represent distances between *ṭhāṭs* accurately, we need at most six dimensions (the fact that NICM uses relative pitch means that we lose a dimension—a *ṭhāṭ* is really an equivalence class modulo the choice of the pitch Sa). Because all *ṭhāṭs* include the pitch Pa, five dimensions suffice, but the number of dimensions is still too great for us to draw a satisfactory representation.

However, we can exploit a feature of *ṭhāṭs* here. As with Arab modes (see [5]), each *ṭhāṭ* is traditionally considered to be formed from two scalar tetrachords. The lower tetrachord begins with Sa and ends with Ma (or Ma *tīvra*) and the upper tetrachord begins with Pa and ends with high Sa. This decomposition suggests a different way of constructing the lattice of theoretical *ṭhāṭs*. First, we note that representing tetrachords, modulo translation, requires only three dimensions; in Figure 3 (left), we locate the lower and upper tetrachords on disjoint lattices, where two tetrachords are adjacent if and only if they differ by one semitone. The product of the two tetrachord graphs (Figure 3, right) can be visualized as two nested tori, each corresponding to a different position of Ma. In this picture, each torus has been cut open to form a large square. (This explains why the *ṭhāṭs* on the left-hand edge are duplicated on the right-hand edge and the *ṭhāṭs* on the bottom edge are duplicated at the top.) *Ṭhāṭs* with the same first three notes appear in the same vertical plane, while *ṭhāṭs* with the same upper tetrachord are in the same horizontal plane. If the edge faces are connected, the resulting graph is isomorphic to the graph of theoretical *ṭhāṭs* (Figure 2).

The construction of Figure 3 was first proposed as a tool for representing modulatory relationships between Arab modes, or *maqāmāt* [5]. Figure 4 contrasts the *ṭhāṭs* of NICM, the Glarean modes, and the Arab modes. (Since Arab musicians use a quarter-tone scale, there are intermediate modes between lattice points. Only about two-thirds of Arab modes are representable on this lattice—some do not repeat at the octave, and others have a different fifth scale degree.) As previously noted, Glarean modes are a subset of the Circle of *Ṭhāṭs*. However, there is surprisingly little overlap between the North Indian and Arab modes. In particular, the Arab system uses the diatonic scale sparingly, preferring instead some scales that divide the octave more evenly (this is possible using quarter tones) and others quite a bit less evenly. The fact that the Circle of *Ṭhāṭs* lies on or near the diagonal of the squares reflects a preference in NICM for what Jairazbhoy calls "balanced" *ṭhāṭs*—*ṭhāṭs* whose upper and lower tetrachords contain roughly the same scalar intervals.

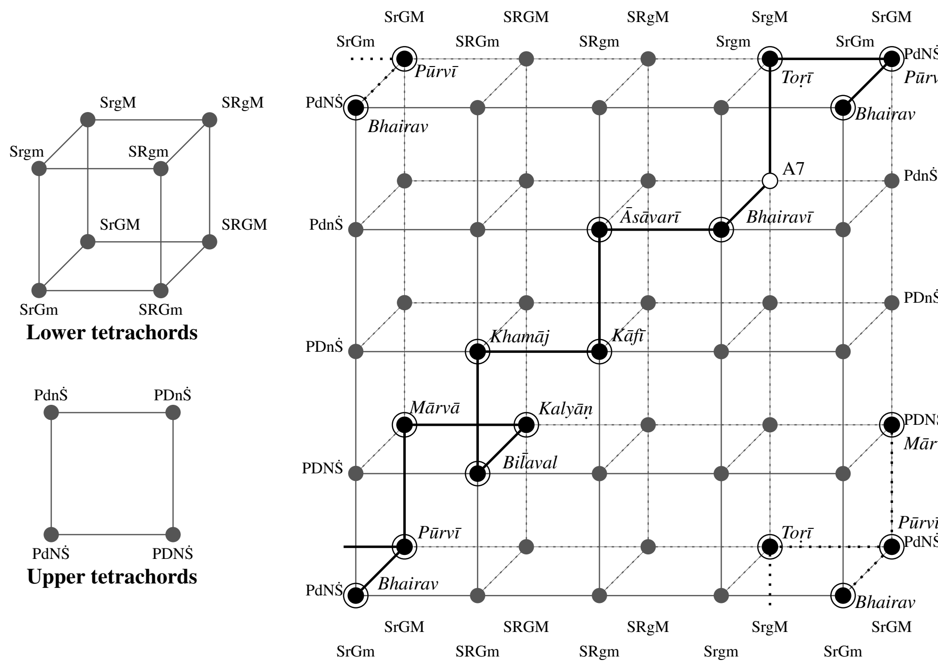

Figure 3: *The lattice of thirty-two theoretical ṭhāṭs, configured as nested tori.*

3. Examples

Although we have discussed the difficulty of identifying *rāgs* with *ṭhāṭs* before, let us see how geometrical methods can help (or at least give us a better visualization). In Figure 5, we contrast *rāg Āsāvarī* with *rāg Mālkauns*. *Rāg Āsāvarī* belongs to *ṭhāṭ Āsāvarī* (indicated by a dotted sphere). Its *āroh* is a subset of four theoretical *ṭhāṭs* and its *pakaṛ* is a subset of two. (Because the graph is a torus, there appear to be six markers for the *āroh*—two of them are repeats.) However, its *avroh* contains exactly the notes of *ṭhāṭ Āsāvarī*. In this situation, there is no ambiguity in the classification of the *rāg*. In contrast, the pentatonic (missing Re and Pa) *rāg Mālkauns* is classified under *ṭhāṭ Bhairavī* (indicated by a dotted sphere). However, due to the omission of Re, its *āroh*, *avroh*, and *pakaṛ* are subsets of two *ṭhāṭs*: *Āsāvarī* and *Bhairavī*. This ambiguity agrees with Bhātkhaṇḍe's aforementioned comment that theorists differ on whether to assign *rāg Mālkauns* to *Bhairavī ṭhāṭ* or to *Āsāvarī ṭhāṭ* [2, vol. 3, p. 701]. Figure 6 depicts two *rāgs* that have "too many" notes. As previously noted, *Kedar* contains both sharp and natural versions of Ma; *Hamir* has this same feature. At present, our models do not distinguish between superset and subset relations: both *rāgs*' *avroh* have *ṭhāṭs Kalyāṇ* and *Biḷāval* as subsets, rather than supersets.

Our models clearly reflect the fact that the relationship between a *rāg* and its *ṭhāṭ* is sometimes ambiguous. In terms of pitch class content, *rāgs* belonging to the same *ṭhāṭ* vary in the degree to which they signal their parent *ṭhāṭ* and the degree to which they resemble each other. Moreover, a *rāg*'s *āroh*, *avroh*, and *pakaṛ* may convey different (and occasionally conflicting) information. However, there are many features of *rāgs* that are not captured by this geometrical representation. Further analysis is needed to determine which features are most predictive of the assignments of *rāgs* to *ṭhāṭs*.

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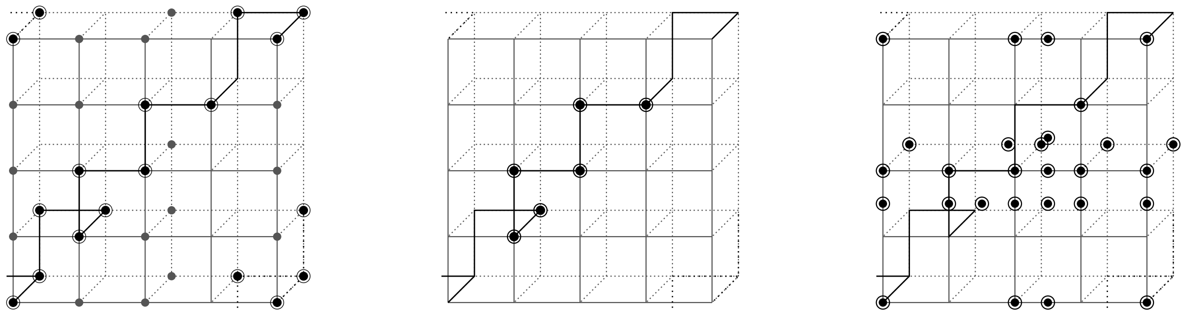

Figure 4: *North Indian ṭhāṭs (left), Glarean modes (center), and Arab modes (right).*

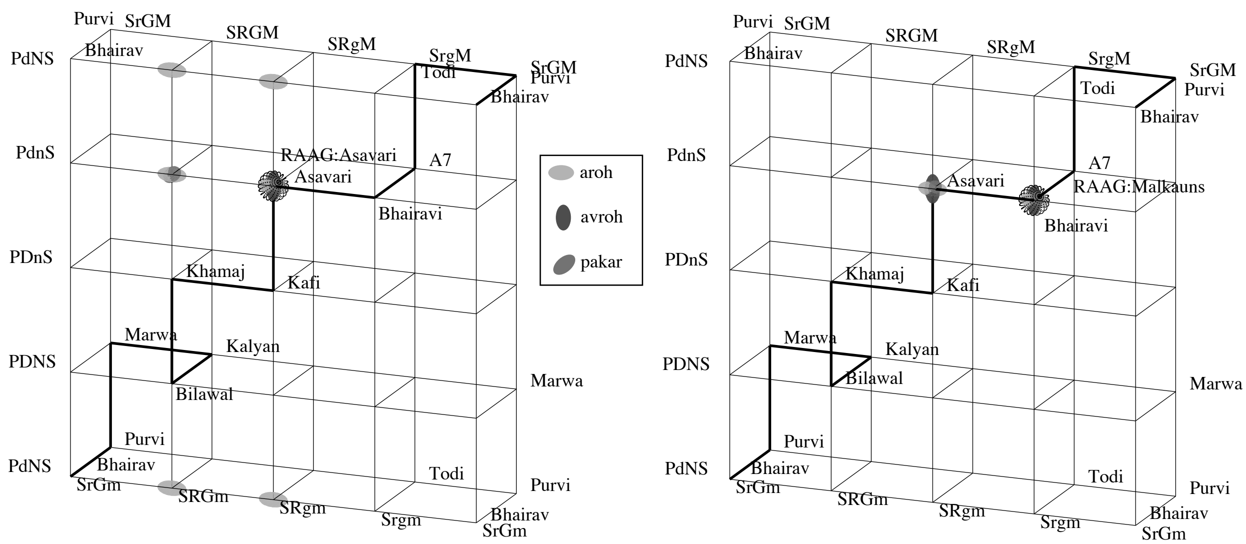

Figure 5: *Rāg Āsāvarī and rāg Mālkauns* (graph generated by MATLAB).

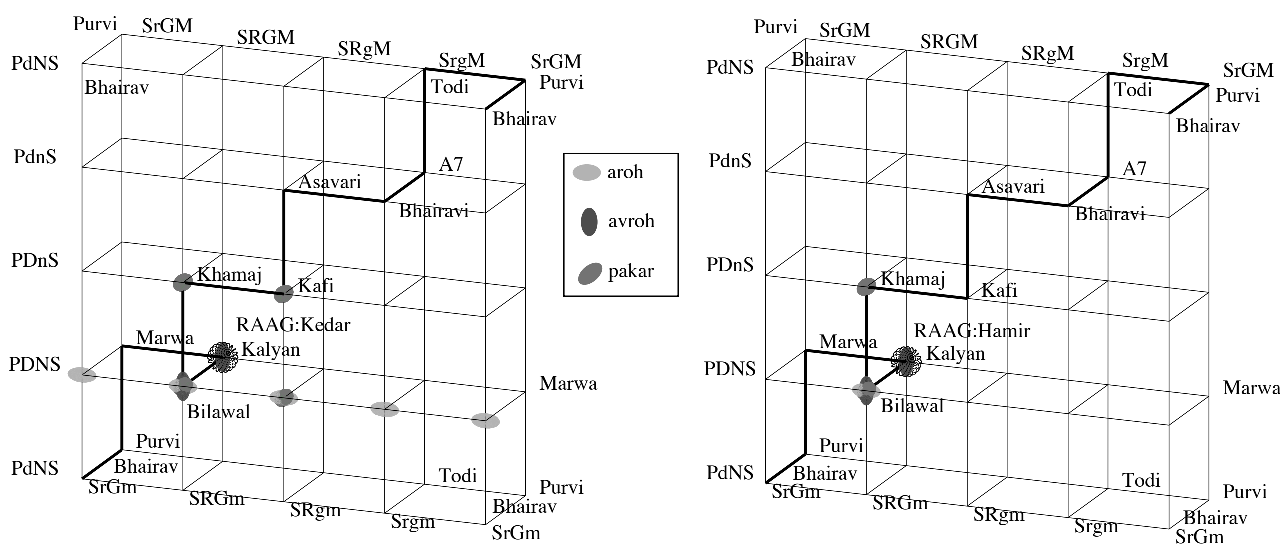

Figure 6: *Rāg Kedar and rāg Hamir* (graph generated by MATLAB).