Geometrical Representations of North Indian Ṭhāṇs and Rāgs

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Abstract
In his seminal works on North Indian classical music theory, V. N. Bhāṭkhāṇḍe (1951, 1954) classified about two hundred rāgs (fundamental melodic entities) by their seven-note parent modes known as ṭhāṇs. However, assigning rāgs to ṭhāṇs is not a straight-forward task. Each rāg is defined by a collection of melodic features that guide a performer’s improvisation. Although these features sometimes point to a unique ṭhāti, in other situations they either give incomplete information (too few notes) or give conflicting information (too many notes). Our goal in this paper is to construct geometrical models that help us to better understand the relationship between ṭhāṇs and rāgs. Following the principles of geometrical music theory (Callender, Quinn, and Tymoczko 2008), we locate the thirty-two “theoretical ṭhāṇs” in a five-dimensional lattice. Jairazbhoy’s “Circle of ṭhāṇs” connecting common ṭhāṇs embeds within this lattice (Jairazbhoy 1971). For a given rāg, our geometrical representations show which theoretical ṭhāṇs contain the notes used in the rāg’s various melodic components separately. We have written MATLAB code that produces images of a database containing a number of rāgs. Our models reveal graphically some of the problematic aspects of Bhāṭkhāṇḍe’s rāg classification system.

1. Introduction

Rāgs are the fundamental melodic entities of North Indian Classical Music (NICM). Rather than being a fixed “tune,” each rāg is a collection of musical features that guide a performer’s improvisation. In his foundational books on North Indian music theory [1, 2], V. N. Bhāṭkhāṇḍe classified rāgs by seven-note modes known as ṭhāṇs (a mode, such as the major or minor mode in Western music, is a scale with a distinguished tonic). While there are close to two hundred rāgs, Bhāṭkhāṇḍe assigns each rāg to one of ten ṭhāṇs. Although this assignment is straightforward in some cases, quite a few rāgs have either too many or too few distinct notes to correspond with a unique ṭhāti.

Our goal in this paper is to construct geometrical models representing set theoretic relationships between ṭhāṇs and rāgs. While scholars have experimented for centuries with geometrical models for Western modes, including circles of major and minor modes and the Neo-Riemannian tonnetz, geometrical models representing the elements of NICM appeared relatively recently, chiefly in the work of Jairazbhoy [6]. Following the principles of geometrical music theory [3], we locate the thirty-two “theoretical ṭhāṇs” in a five-dimensional lattice. For a given rāg, our geometrical representations show which theoretical ṭhāṇs are supersets of notes used in the rāg’s āroh, avroh, and pakar separately. These reflect the degree to which a rāg is unambiguously identified with its ṭhāṇ. We have written MATLAB code that produces images of a database of rāgs.

The basics of North Indian music theory are as follows. As in Western theory, seven notes, Sa, Re, Ga, Ma, Pa, Dha, and Ni span an octave; this sequence of notes repeats in higher and lower octaves. Of these notes, Re, Ga, Ma, Dha, and Ni have two positions, śuddha (natural) and vikrit (altered), which may either be komal (flat) or tīvra (sharp). The only note among these to have a āvīrt position is Ma, while the rest have komal and śuddha versions. Thus the twelve notes in an octave, successively a semitone apart, are: Sa, Re (komal), Re (śuddha), Ga (komal), Ga (śuddha), Ma (śuddha), Ma (tīvra), Pa, Dha (komal), Dha (śuddha), Ni (komal), Ni (śuddha). We will use the abbreviated list \{S, r, R, g, G, m, M, P, d, D, n, N\} when convenient. We note that Indian note names indicate relative, rather than absolute, pitch; the performer is free to choose the actual pitch identified as “Sa.”

A ṭhāti is an ordered collection of the seven notes, where only one version of each note may be selected. Since five of the notes have two positions, it is theoretically possible to create thirty-two (2⁵) ṭhāṇs. However, only the ten ṭhāṇs listed in Table 1 are commonly used in NICM. Six of these, including the major (Ionian) and minor (Aeolian) modes, are known in the West as Glarean modes.¹

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²Glarean modes, named for the sixteenth century music theorist Heinrich Glarean, all belong to the same set class, meaning that, modulo cyclic permutation or reversal, they have the same sequence of intervals between adjacent notes. This set class, the diatonic scale, has quite a few desirable properties, including the fact that it is nearer than any other seven-note collection in twelve-tone equal temperament to the even division of an octave into seven parts. In addition, it is “generated” by a sequence of six perfect fifths modulo the octave (see [4]).

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While there are only ten thāts in common use, there are about two hundred rāgs. A rāg is a melodic theme upon which a performer improvises while staying within the allowable boundaries of note patterns and combinations specific to that rāg. Each rāg is characterized by its ascending and descending sequences (āroh and avroh), its “catch phrase” (pakara), its emphasized notes (vādi and samvādi), the number of notes it contains (jāti), the octave emphasized, and the time of day it is performed. Rāgs may be pentatonic, hexatonic, or heptatonic depending on the number of distinct notes they use.

Table 2 summarizes five rāgs from three thāts. In theory, a rāg is assigned to a parent thāt largely on the basis of agreement of notes in the rāg with those of the thāt. This is clearly true in the case of rāg Āsāvari: the union of the set of notes in its āroh, avroh, and pakara corresponds exactly with Āsāvari thāt. Although both its āroh and pakara are “incomplete” in that they contain less than seven distinct notes, Āsāvari is the only one of the ten common thāts that have the rāg’s āroh or pakara as subsets. (For example, its āroh contains the notes S, R, m, P, and d. Four of the thirty-two theoretical thāts also contain these notes; of them, only Āsāvari is a common thāt.)

Identifying rāgs with thāts based on subset relationships is not always straightforward. In particular, the notes of hexatonic and pentatonic rāgs are subsets of more than one thāt. Bhattacharya mentions these difficulties in his major work, the Kramik Pustak Mālikā [2], where he provides brief descriptions for each of about 180 rāgs. For example, rāg Mālkauṇs is a pentatonic rāg containing only the notes {S, g, m, d, n}. On the basis of notes alone, it could equally well belong to Āsāvari or Bhauravī. Bhattacharya notes that rāg Mālkauṇs “is generated from Bhauravī thāt . . . some say that it is in Āsāvari thāt” [2, vol. 3, p. 701, translated from Hindi].

The comparison of rāgs Shuddhakalīyān and Bhupālī reveals another challenge for the practitioner of NICM. Bhattacharya singles out certain rāgs that are “close” and explains what a performer must do to avoid crossing over to a neighboring rāg. For example, while Shuddhakalīyān is similar to Bhupālī, but, “unlike Bhupali, in this rāg the lower octave is used more . . . In avroh [the note] Ni is used many times and this distinguishes it from Bhupali” [2, vol. 4, pp. 60-61]. Note that Shuddhakalīyān’s heptatonic avroh not only distinguishes it from Bhupālī but also identifies the thāt. (In general, we note that a rāg’s avroh is more likely than its āroh or pakara to signal its thāt.)

In contrast, rāg Kedar has “too many” distinct notes (eight) rather than too few. It belongs to Kalyān thāt, even though subset analysis seems to suggest Bilavāl (in particular, its āroh belongs to Bilavāl, its avroh contains both Bilavāl and Kalyān, and its pakara belongs to Bilavāl, Khamāj, or Kāfi). Moreover, Kedar’s vādi (emphasized note) is a natural Ma, while Kalyān thāt has a sharp Ma. Bhattacharya comments that both sharp and natural forms of Ma are used. Ancient writers did not allow use of sharp Ma in Kedar and considered it to be under thāt Bilavāl. Presumably, the sharp Ma “trumps” the natural.
Due to the association between rāgs and times of day, the depiction of rāgs on a circle is natural. Rāgs belonging to the same thāts are typically performed either at the same time or separated by half a day. On this basis, Bhāṭkhānde proposed to identify thāts with times of day on a twelve-hour cycle. Jairazbhoy [6, p. 63] took the logical next step by arranging thāts on a circle according to Bhāṭkhānde’s time theory, as in Figure 1 (left). Remarkably, nine of the ten thāts, starting with Bhairav and proceeding clockwise to Bhairavi, form a sequence in which each thāt is related to its neighbors by a one-semitone alteration in one of its notes. For example, we move from Kalyān to Bilaval by changing Kalyān’s sharp Ma to a natural Ma (F→F♯), while we move from Kalyān back to Mārvā by flatting Kalyān’s natural Re (D→D♭). In other words, with the exception of Torī, thāts that are adjacent in time are linked by voice leadings (bijections between collections of notes) which are “efficient” in that only a small amount of chromatic alteration takes place. Roy [7, p. 82] theorizes that the agreement between the ordering of thāts based on efficient voice leading and the ordering based on time theory is probably due to the “tendency of rāgas to follow the line of least resistance in the easy transition from scale to scale . . . observed to a certain extent by all musicians.” Since moving from one thāt to another requires retuning some musical instruments, it is advantageous to arrange the cycle so any two neighboring thāts share as many common tones as possible. In the sequence of six thāts from Kalyān to Bhairavi, one has the added advantage that the new pitch is always a perfect fifth from one of the notes in the original scale. After the octave, the perfect fifth is the easiest interval to tune.) We also note that typical models of Western modes share the feature that the modes are linked by efficient voice leading [3].

Is there a way the voice leading approach can be made to include Torī? And what of the thirty-two theoretical thāts: can they be incorporated into a model? We locate theoretical thāts as vertices of a graph in Figure 2 (Jairazbhoy depicts an isomorphic graph in [6, p. 184]). Two thāts are connected by an edge if and only if they differ by one semitone. Note that, although the graph is a convenient model for local connections between thāts, it does not represent distances—each edge in the graph represents a one-semitone alteration, but the edges are different lengths. Moreover, it does not represent all possible pathways between thāts.

Bhāṭkhānde’s ten common thāts, indicated by ringed circles, define a connected subgraph of the lattice. In order to complete a cycle, Jairazbhoy adds a theoretical thāt labelled “A7” (so called because of his classification scheme). This move successfully incorporates Torī but leaves out Bhairav. Jairazbhoy’s “Circle of thāts,” as in Figure 1 (right), embeds as a cycle in the graph of theoretical thāts. The graph also reveals the problem: Pārvī, Bhairav, Torī, and Bhairavi lie on the vertices of a cube in the lattice, and there is no path that connects them all, using transitions where some note is altered by a single semitone. An alternate to Jairazbhoy’s solution is to allow the path to bifurcate, connecting Pārvī to both Torī and Bhairav, then connecting Torī, Bhairav, and Bhairavi to the unique theoretical thāt that is within a one-semitone alteration of all of them. (Jairazbhoy [6, p. 97-99] cites historical and theoretical reasons for preferring “A7” to this thāt, however.)
Geometrical music theory provides a way of thinking about geometrical representation in general (see [3]): any musical object that can be represented by an $n$-tuple of pitches corresponds to a point in some $n$-dimensional Euclidean space. Equivalence relations, such as octave equivalence, define quotient maps on Euclidean space producing a family of singular, non-Euclidean, quotient spaces—orbifolds. Points in these spaces represent equivalence classes of collections of notes, such as chords or scales. Any voice leading corresponds to a line segment or path in an orbifold. In order to represent distances between thāts accurately, we need at most six dimensions (the fact that NICM uses relative pitch means that we lose a dimension—a thāt is really an equivalence class modulo the choice of the pitch Sa). Because all thāts include the pitch Pa, five dimensions suffice, but the number of dimensions is still too great for us to draw a satisfactory representation.

However, we can exploit a feature of thāts here. As with Arab modes (see [5]), each thāt is traditionally considered to be formed from two scalar tetrachords. The lower tetrachord begins with Sa and ends with Ma (or Ma tīvra) and the upper tetrachord begins with Pa and ends with high Sa. This decomposition suggests a different way of constructing the lattice of theoretical thāts. First, we note that representing tetrachords, modulo translation, requires only three dimensions; in Figure 3 (left), we locate the lower and upper tetrachords on disjoint lattices, where two tetrachords are adjacent if and only if they differ by one semitone. The product of the two tetrachord graphs (Figure 3, right) can be visualized as two nested tori, each corresponding to a different position of Ma. In this picture, each torus has been cut open to form a large square. (This explains why the thāts on the left-hand edge are duplicated on the right-hand edge and the thāts on the bottom edge are duplicated at the top.) Thāts with the same first three notes appear in the same vertical plane, while thāts with the same upper tetrachord are in the same horizontal plane. If the edge faces are connected, the resulting graph is isomorphic to the graph of theoretical thāts (Figure 2).

The construction of Figure 3 was first proposed as a tool for representing modulatory relationships between Arab modes, or maqāmāt [5]. Figure 4 contrasts the thāts of NICM, the Glarean modes, and the Arab modes. (Since Arab musicians use a quarter-tone scale, there are intermediate modes between lattice points. Only about two-thirds of Arab modes are representable on this lattice—some do not repeat at the octave, and others have a different fifth scale degree.) As previously noted, Glarean modes are a subset of the Circle of Thāts. However, there is surprisingly little overlap between the North Indian and Arab modes. In particular, the Arab system uses the diatonic scale sparingly, preferring instead some scales that divide the octave more evenly (this is possible using quarter tones) and others quite a bit less evenly. The fact that the Circle of Thāts lies on or near the diagonal of the squares reflects a preference in NICM for what Jairazbhoy calls “balanced” thāts—thāts whose upper and lower tetrachords contain roughly the same scalar intervals.
3. Examples

Although we have discussed the difficulty of identifying rāgs with thāts before, let us see how geometrical methods can help (or at least give us a better visualization). In Figure 5, we contrast rāg Āsāvari with rāg Mālkauns. Rāg Āsāvari belongs to thāt Āsāvari (indicated by a dotted sphere). Its āroh is a subset of four theoretical thāts and its pakar is a subset of two. (Because the graph is a torus, there appear to be six markers for the āroh—two of them are repeats.) However, its avroh contains exactly the notes of thāt Āsāvari. In this situation, there is no ambiguity in the classification of the rāg. In contrast, the pentatonic (missing Re and Pa) rāg Mālkauns is classified under thāt Bhairavī (indicated by a dotted sphere). However, due to the omission of Re, its āroh, avroh, and pakar are subsets of two thāts: Āsāvari and Bhairavī. This ambiguity agrees with Bhātkhande’s aforementioned comment that theorists differ on whether to assign rāg Mālkauns to Bhairavī thāt or to Āsāvari thāt [2, vol. 3, p. 701]. Figure 6 depicts two rāgs that have “too many” notes. As previously noted, Kedar contains both sharp and natural versions of Ma; Hamir has this same feature. At present, our models do not distinguish between superset and subset relations: both rāgs’ avroh have thāts Kalyāṇ and Bilaval as subsets, rather than supersets.

Our models clearly reflect the fact that the relationship between a rāg and its thāt is sometimes ambiguous. In terms of pitch class content, rāgs belonging to the same thāt vary in the degree to which they signal their parent thāt and the degree to which they resemble each other. Moreover, a rāg’s āroh, avroh, and pakar may convey different (and occasionally conflicting) information. However, there are many features of rāgs that are not captured by this geometrical representation. Further analysis is needed to determine which features are most predictive of the assignments of rāgs to thāts.

References

Figure 4: North Indian thāts (left), Glarean modes (center), and Arab modes (right).

Figure 5: Rāg Āsāvari and rāg Mālkauns (graph generated by MATLAB).

Figure 6: Rāg Kedar and rāg Hamir (graph generated by MATLAB).