Brunnian Weavings

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Abstract

In this paper, we weave *Borromean Rings* to create interesting objects with large crossing number while retaining the characteristic property of the Borromean Rings. Borromean Rings are interesting because they consist of three rings linked together and yet when any single ring is removed the other two rings become unlinked. The first weaving applies an iterative self-similar technique to produce an artistically interesting weaving of three rings into a fractal pattern. The second weaving uses an iterative *Peano Curve* technique to produce a tight weaving over the surface of a sphere. The third weaving produces a tight weaving of four rings over the surface of a torus. All three weavings can produce links with an arbitrarily large crossing number. The first two procedures produce Brunnian Links which are links that retain the characteristic property of the Borromean Rings. The third produces a link that retains some of the characteristics Borromean Rings when perceived from the surface of a torus.

1. Borromean Rings and Brunnian Links

Our goal is to create interesting ***Brunnian Weavings*** with an arbitrarily large crossing number. The thought in this paper is that Borromean Rings become more interesting as they become more intertwined. ***Borromean Rings*** consist of three rings linked together and yet when any single ring is removed the other two rings become unlinked. Figure 1.1 shows the most common representation of the Borromean Rings. This name comes from their use in the Borromeos' family crest in the fifteenth century. Although Peter Tait, in 1876, was the first mathematician to study these rings, the name Borromean Rings was not used until 1962 in a paper by Ralph Fox. See [5] for an excellent discussion on the history of Borromean Rings, which, not surprisingly, includes examples that precede their use by the Borromeo family.

The Borromean Rings consist of three linked rings which are pair-wise unlinked. We can see that they are pair-wise unlinked by noting that the light gray curve always crosses over the medium gray curve, the medium gray curve always crosses over the dark gray and the dark gray curve always crosses over the light gray. The Borromean Rings is the simplest example of a Brunnian Link. A ***Brunnian Link*** is *a collection of loops linked together such than if any one loop is cut and removed the other loops separate into unknotted components.*

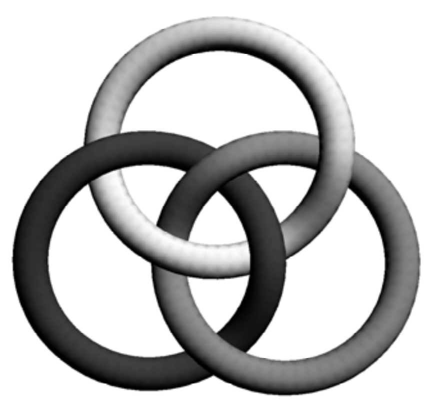

Figure 1.1: *Borromean Rings*

Throughout this paper, we make sure that at each step of our process our links retain this Brunnian Property by following two guiding rules of construction that we observe in Figure 1.1. The first rule restricts how loops cross each other. The second rule restricts how loops cross themselves. Three threads can cross each other in six different over/under combinations as shown in Figure 1.2. Our first rule of construction is to only allow different loops to cross over each other according to the crossings on the left of Figure 1.2. That is:

light gray > medium gray > dark gray > light gray

where ">" means ***crosses above***. Second, we require that each individual component of the link, when ignoring the other components, remains unknotted. Notice that the crossing ordering is non-transitive. Note that non-transitivity is expected since knowledge about two crossings ordinarily does not imply anything about the adjacent crossings. Our first construction in this paper will remain consistent with the second rule by only adding simple twists. For the other two constructions we remain consistent with this rule by not having any self-intersections. Note that for both rules the concept of *crossings* is only valid when the link is projected onto some two-dimensional surface. For our first construction, we project onto the plane. For our second, we project onto the sphere. For the third, we project onto the torus. The first two projections are topologically equivalent but the third case is not.

Figure 1.2: *The Six Crossing Types*

2. A Self-Similar Weaving of the Borromean Rings

In this section we weave the standard planar projection of the Borromean Rings by cutting and pasting small copies of the Borromean Rings into a larger Borromean Ring. This construction was inspired by Robert Fathauer's graphic design *Infinity*[4] presented in his paper *Fractal Knots Created by Iterative Substitution*[3]. The Borromean Rings in Figure 2.1 contain seven regions consisting of three distinct types: A, B, and C. In this paper we shall place small copies of the Borromean Rings into all of the regions of type A. We shall then place tiny Borromean Rings into all the regions of type A in the small rings. This process can be continued an arbitrary number of times. Applying this process to region B is less interesting. It turns out that this process fails when applied to regions of type C.

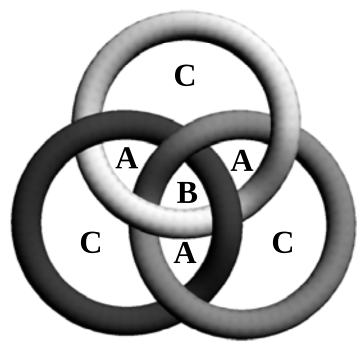

Figure 2.1: *Regions*

Since we glue rings of like colors, once we shrink the original image, we will need to flip the image so that points P_1, P_2 and P_3, in Figure 2.2, on a small copy of the rings line up with points Q_1, Q_2 and Q_3 on a large copy of the rings. To have the rings smoothly transition from the large to the small, we need to have the ring thinner at all of the Q's and thicker at all of the P's. For this particular overlap of rings, we need to shrink the rings by a factor of 1/5 and so the initial ring needs to be 5 times thicker at the P's than at the Q's as shown in Figure 2.3. We are now ready to cut three small copies of Figure 2.3 at the P's and cut a large copy of Figure 2.3 at the Q's and carefully paste together as shown in Figure 2.4. Although this process can be repeated indefinitely, only one or two more iterations are practical. Figure 2.5 shows a perspective view of the rings after two iterations.

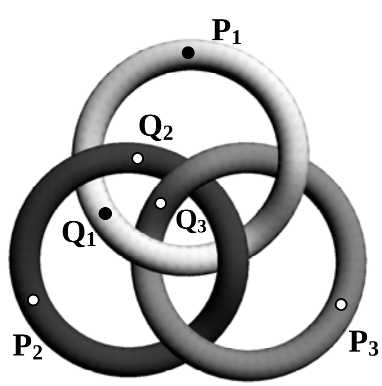

Figure 2.2: *Attachment Points*

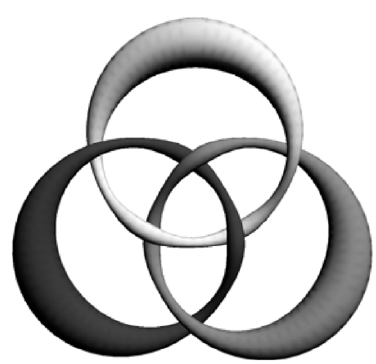

Figure 2.3: *Adjusted for Thickness*

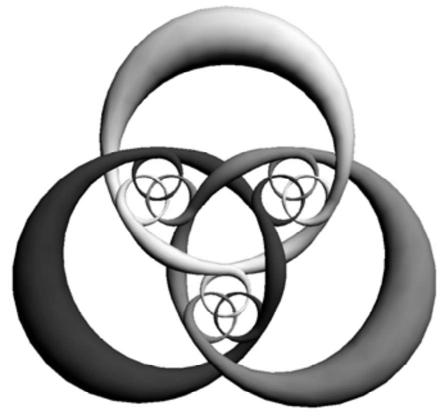

Figure 2.4: *First Iteration*

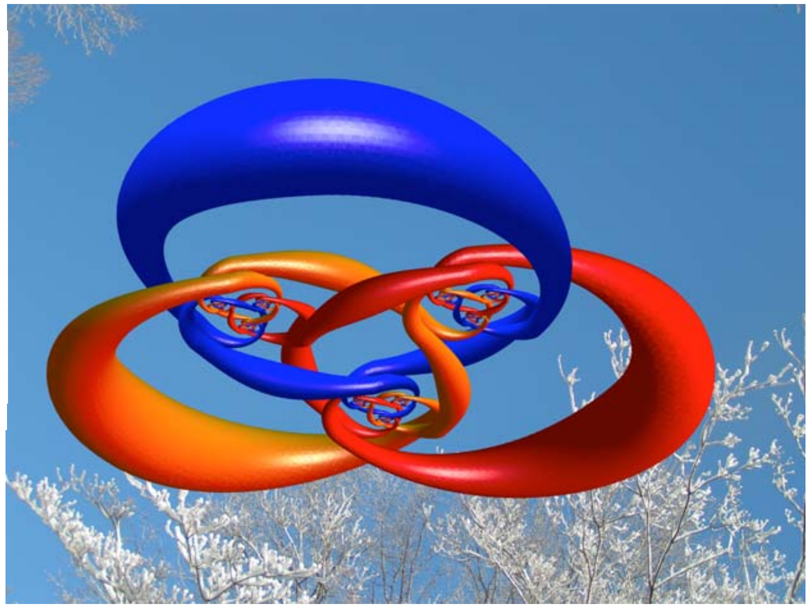

Figure 2.5: *A Planar Brunnian Weaving*

The rings shown in these Figures 2.4 and 2.5 are Brunnian Links since they follow the two rules listed above. When any one ring is cut and removed, the other two rings separate into two distinct layers due to the crossing rule in Figure 1.2. Once these two rings separate into layers, we can repeatedly untwist the smallest rings in each component until each ring is unknotted.

Now observe that if we tried to apply a similar procedure in region C then point P_1 on the smaller copy would attach to point P_1 on the larger copy. With careful distortions of the rings, this could be accomplished. However, the next iteration must also attach at the same point. Thus, if the top left side of the top ring was attached to the top right side of the medium size ring then medium ring would attach to a tiny ring. But then this tiny ring would have to reattach back to the top right of the large ring! In general, any point, such as the P's, that is used to attach to a larger ring can not be used to attach to a smaller ring. This prevents an iterative replace procedure of the Borromean Rings in region C for this projection.

3. A Spherical Brunnian Weaving with Three Threads

In this section we increase the level of complexity by creating a *dense* weaving which retains the Brunnian Property. Since our construction will look more like weaving with yarn we shall call each component a *thread* instead of a ring. Our goal is to create a *tight* weave meaning that we can apriori set a size so that any region of this "small" size contains all three threads. To achieve this we use a technique similar to Peano Curves to iteratively replace each region with a more complex region so that each segment of thread is replaced by a longer and thinner segment of thread that intertwines with threads on its right and intertwines with threads on its left. Unlike Peano Curves, we do not take this to the limit. The projection of the Borromean Rings in Figure 1.1 has a boundary which presents an extra level of difficulty. To avoid a boundary we rearrange the Borromean Rings to encompass a sphere, as shown in Figure 3.1.

Figure 3.1: *Borromean Rings Encompassing a Sphere*

Before we begin replacing regions, let us observe three properties of the initial configuration. First, observe that by projecting the threads of Figure 3.1 onto the enclosed sphere, so that "down" means towards the center of the sphere, we retain the crossing rules from Figure 1.2 that the dark gray thread is above the light gray thread which is above the medium gray thread which is above the dark gray. Second, observe that the rings partition the surface of the sphere into eight triangular regions. Third, notice that when we allow one color to be mapped onto another, we have rotational symmetry about any diameter line which passes through the center of any opposite pair of triangles. Finally, notice that relative to the projection onto the sphere, none of the rings have self-crossings.

A Triangular Weaving. Our first substitution rule is to replace every triangle of the form ABC, shown in the left side of Figure 3.2, with the weaving shown on the right side of Figure 3.2. Notice that the dark thread along side AB on the original triangle is replaced by a dark thread that starts at B but exits the object at the midpoint *m* of side AB. Looking at the midpoint is a simple way to determine the correct coloring of the threads. Notice that both the triangle on the left and the weaving on the right use only the *valid* crossing stated in Figure 1.2: *light gray* crossing above *medium gray* crossing above *dark gray* crossing above *light gray.* This is the simplest weaving that satisfies our crossing property.

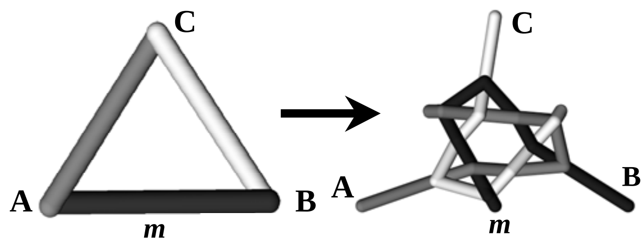

Figure 3.2: *Triangle Substitution*
Initial Triangle *Triangular Weaving*

Now let us consider a pair of triangles attached along side AB. If we rotate Figure 3.2b about point *m* then the dark thread would align correctly, but the colors would be off at B. In the original triangle, B is the intersection of the dark and light threads, but by rotating this weaving, the medium gray is now attaching to point B. On the other hand, if we flip the weaving as a mirror image across the line AB, then the dark thread starts at B and loops right back to B. Of these, we cannot have a thread looping back. Thus, we must rotate the object and then recolor in the opposite color pattern changing from dark-light-medium to dark-medium-light when reading counterclockwise. We must also flip the crossing on the new triangle to remain consistent with our crossing rules.

To check the validity of this substitution, we apply it to the pair of triangles shown on the left of Figure 3.3. We can now see that the dark thread starts at A and ends at B after weaving throughout both triangles. In general, when tracing a thread in either direction, the thread weaves first into the triangle on the threads right-hand side and then into the triangle on its left-hand side. We can call the top triangle an *over triangle* and the bottom triangle an *under triangle* reading the weavings in a counterclockwise fashion. Since every vertex of a weaving has degree four, we can always *color* all of the faces of any weaving with two colors in a checkerboard fashion. Here we can call these "colors" *over* and *under*. This demonstrates that the *pair wise* replacement pattern is valid for the entire object since every adjacent pair of triangles contains both an *over triangle* and an *under triangle*.

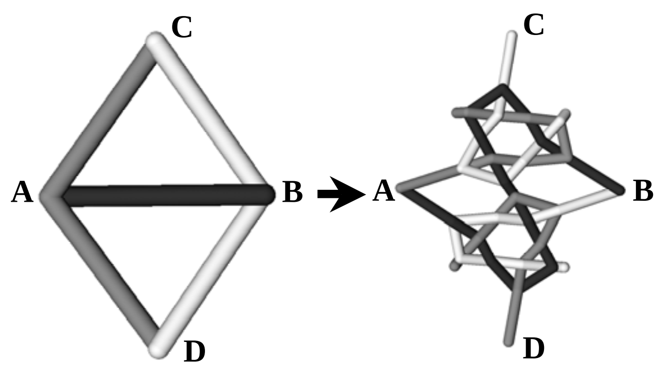

Figure 3.3: *Substitution for a Pair of Triangles*

The First Iteration. Applying this substitution step to the eight triangles in the Borromean Rings in Figure 3.1, we obtain the weaving shown in Figure 3.4a. All four views in Figures 3.4 and 3.5 are from the center of a triangle in the "Northern Hemisphere". As can be seen in Figure 3.4b, the removal of the dark gray thread unlinks the light gray thread from the medium gray thread so that the light gray could be expanded outwards to be “above” or outside a sphere which contains the medium gray. Moreover, by construction and as can be seen in Figure 3.5, each individual thread has no self-crossing relative to the sphere and so each thread is unknotted. Similarly, the other threads are unlinked when a thread is removed demonstrating that this weaving is a Brunnian Link.

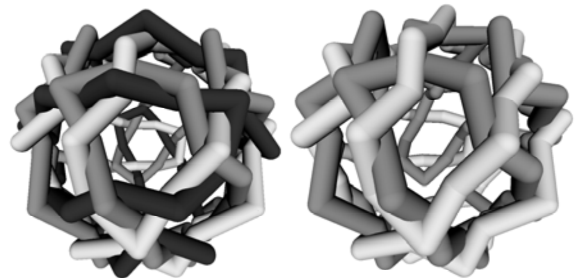

Figure 3.4:
A Spherical Weaving *Light above Medium*

Symmetry. Since Figure 3.1 has rotational symmetry about the center of any triangle, and since Figure 3.2 has rotational symmetry about its center, so does the final Spherical Weaving in Figure 3.4a. A rotation of 120° along the axis of the viewpoint rotates each thread onto the next color. We can see this rotation best by comparing Figures 3.5a, the equatorial thread that weaves up and down into the Northern and Southern Hemisphere, and Figure 3.5b, the thread that loops over the North and South Pole.

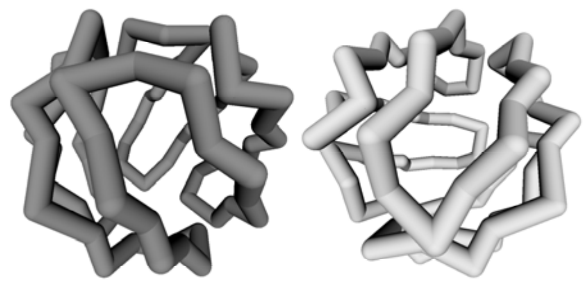

Figure 3.5: *Rotational Symmetry*
The Equatorial Thread *A North Pole Thread*

A Hexagonal Weaving. To extend the weaving to the next level we must create a replacement procedure for hexagons that is consistent with our crossing rules and is consistent with the triangular weaving. Figure 3.6 shows an outside hexagon (thin) with a replacement weaving (thick). Consistent with triangles, each thread first weaves into the face on its right side, returns at the midpoint of the side and then weaves into the face on its left side. We see in Figure 3.7 that the triangles and the hexagons have the same half hexagon pattern along the edges. Thus, this hexagon substitution is consistent with the triangle substitution given in Figure 3.2 verifying that this hexagon weaving will join with another hexagon weaving or with a triangle weaving and form a pair of hexagons along the common edge as in Figure 3.3. The author conjectures that this is the minimal weaving which will satisfy our requirements.

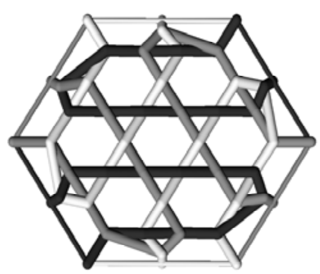

Figure 3.6:
A Hexagon Weaving

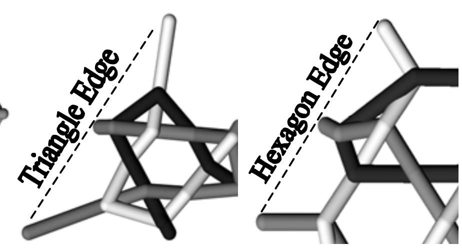

Figure 3.7:
Triangle Edge ↔ *Hexagon Edge*

The Second Iteration. Applying the hexagon replacement rule and the triangle replacement rule to the triangles and all the hexagons in Figure 3.4a we get the Spherical Brunnian Weaving in Figure 3.8. As before, this view is the same view as in Figures 3.4 and 3.5. In Figure 3.9, we see a single thread and a pair of threads as viewed from the North Pole. Observe that a single strand has four-fold symmetry. The weaving only has two-fold symmetry about the North Pole. The complete weaving does not have four-fold symmetry because if we allow the light and the dark to switch then every crossing will be backwards. We can see that a quarter rotation in Figure 3.1 switches the crossings. We can also see that to exchange two colors in our crossing rule causes all of the crossing orders to reverse.

Euler Characteristic. The substitution step introduced hexagonal regions. We can use Euler’s Formula to demonstrate that introducing polygons of higher order is necessary. Euler’s Formula states that the *number of Faces – number of Edges + number of Vertices* = 2 for a sphere. Using an idea from Conway’s book *The Symmetries of Things* [2], we can calculate the characteristic of an individual face. Every edge of the face has a value of ½ since the edge is shared between two faces. Every vertex is of degree four since it represents a crossing of two threads. Thus, every vertex is shared by four faces and has a value of ¼. Thus, our triangular faces have a characteristic of $1 - 3\cdot\frac{1}{2} + 3\cdot\frac{1}{4} = \frac{1}{4}$. Since a sphere has characteristic of 2, any weaving using only triangles must consist of exactly 8 triangles. As we add extra triangles, we must compensate by adding polygons with negative characteristic. A Brunnian weaving must alternate the strands of thread forcing each polygon to be either an *over* or an *under* orientation of the crossings. Therefore, our three colors must cycle around the edge of every face. Thus, the number of edges of every polygon must be a multiple of 3. Here we use triangles and hexagons. It may be possible to use polygons of higher order. Since hexagons have characteristic $1 - 6\cdot\frac{1}{2} + 6\cdot\frac{1}{4} = -\frac{1}{2}$, for every pair of extra triangles we must compensate with one hexagon. In general, for any weaving with hexagons and triangles on the sphere or on the plane: **#Triangles = 2 · #Hexagons + 8.**

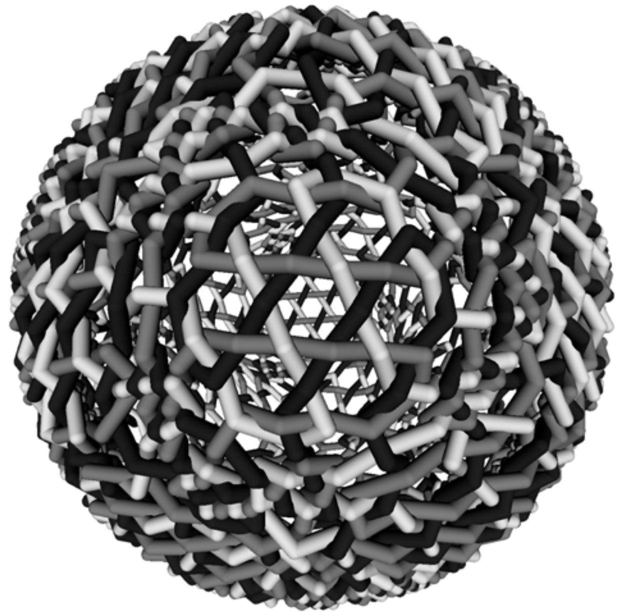

Figure 3.8:
A Three-Thread Spherical Brunnian Weaving with Crossing Number 2,118

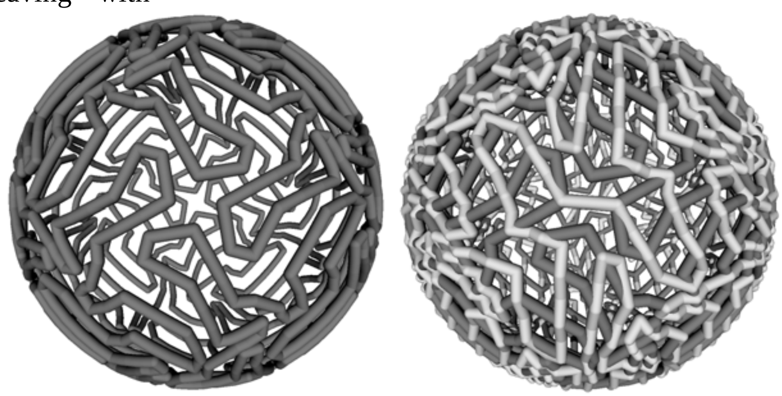

Figure 3.9: *Views from the North Pole*
The Equatorial Thread *Light Gray above the Medium Gray*

Counting Faces. Looking back at Figure 3.3, we see that the triangular weaving produces a pair of hexagons along edge AB. Thus, we can count the faces in the weaving in Figure 3.2 as nine triangles, one hexagon, and six half hexagons. In general, every triangle is replaced by 9 triangles and 4 hexagons. Thus, the Borromean Weaving in Figure 3.4a consists of eight times this number: 72 triangles and 32 hexagons. Counting the faces in Figure 3.6, we see that every hexagon is replaced by 13 hexagons and 24 triangles. Repeatedly applying these formulas counts the number of triangles and hexagons in each iteration of this weaving process. The values are listed in Table 3.10.

The Crossing Number. The crossing number for a link is defined as the *minimum number of crossing* for all possible projections of this link onto a plane. Determining the crossing number is often difficult. In our case, we may use a theorem conjectured by Tait in 1876 and proved in 1987 by Thistlethwaite, Kauffman, and Murasugi: *If a projection of a link onto the plane is reduced and if the crossings alternate*

Weaving Level	Number of Triangles	Number of Hexagons	Crossing Number
0	**8**	**0**	**6**
1	**72**	**32**	**102**
2	**1,416**	**704**	**2,118**
3	**29,640**	**14,816**	**44,454**

Figure 3.10:
Crossing Numbers for Spherical Borromean Weaving

then the crossing number of the link is the crossing number of this particular projection of the link. Although we projected these weavings onto the sphere, topologically this is equivalent to projecting onto the plane. That is, we may enlarge any one face in our weaving, peal the weaving off the sphere and lay it flat while preserving the crossings. Thus, the crossing number of each weaving is simply the number of vertices. To count the vertices we again use Euler's Formula. Since every vertex is attached to four edges and since every edge has two vertices, there must be twice as many edges as vertices. Substituting this into Euler's Formula:

Crossing Number = #Vertices = #Faces – 2 = #Triangles + #Hexagons – 2.

A Tight Weave. Let us assume that we have three Borromean Rings circling the Earth. By looking at the two replacement weavings, we see that at each stage the *diameter* of each polygon shrinks by at least a half at each iteration where diameter means the maximum distance between any pair of vertices in the polygon. After 36 iterations, the diameter of every face in the weaving would be less than 1/200th of an inch. This would be comparable to high quality 400 thread-count linen.

4. A Tori-Brunnian Weaving with Four Threads

Can we weave threads without using triangles? By Euler's Characteristic, triangles are the only polygons with degree 4 vertices with positive characteristic. However, since squares have a characteristic of 0 and tori have characteristic 0, we can tile a torus with as many squares as we wish. But a square weave is just regular fabric! Let us consider the piece of fabric shown in Figure 4.1. Notice that crossings of the thread colors follow the ordering:

Light gray > Medium gray > Dark gray > Black > Light gray

where ">" means "crossing above". This weaving has the property that when any one color is removed the other three colors separate into layers and each layer contains parallel threads of the same color. In Figure 4.2, the dark gray thread has been removed. The weaving now separates into a top layer of black threads, a middle layer of light gray threads, and a bottom layer of medium gray threads.

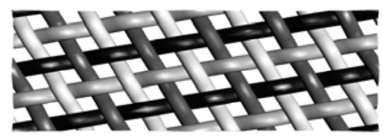

Figure 4.1: *A Fabric Piece with Four Thread Colors*

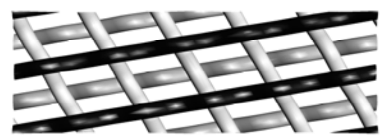

Figure 4.2: *Three Colors Forming Three Layers*

We now want to apply the standard process of identifying opposite edges of Figure 4.1 to roll this piece of fabric into a torus. First, roll the top edge down to meet the bottom edge to form a tube. When we do this, the dark gray threads connect to form a single thread spiraling around our tube. Likewise, the light gray threads connect to form a parallel thread spiraling around our tube. Next, we can pull the two ends of our tube around and connect them to form a torus as shown in Figure 4.3. When we do this, the two loose ends of the dark gray threads join and the two ends of the light gray threads join to each form single loop with no self-intersections relative to the surface of the torus. Likewise, the black thread and the medium gray thread wrap around the torus passing through the hole of the torus once and join to form a second pair of parallel loops of threads. Thus, this weaving is now a link consisting of four components, the different colored threads. Any loop on the surface of a torus with no self-crossings is called a *torus knot.* Thus, each of the four threads are torus-knots. The first two are called (m, 1) since they pass through the hole of the torus m times while making a singe revolution around the torus. The second two threads are called (1, n) since they pass through the center once while revolving n times.

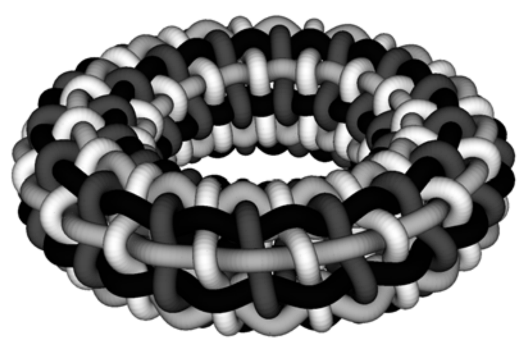

Figure 4.3: *A Tori-Brunnian Weaving using Four Threads*

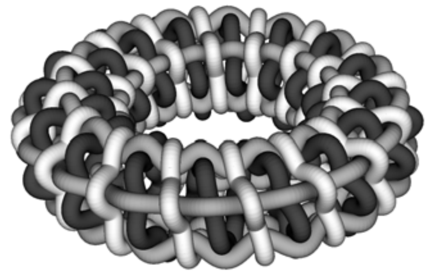

Figure 4.4: *Three Threads Separating into Three Layers*

We project this weaving onto an enclosed torus so that our sense of up and down, and hence of "crossing over," is relative to the local region of the surface. Since every local region of the torus is equivalent to the piece of fabric shown in Figure 4.1, our crossing rules remain valid. Thus, as we see in Figure 4.4, when any one thread is removed the other three threads are *tori-splittable*, they can be separated into layers over the torus. In Figure 4.4, the black thread has been removed and the medium gray thread forms a layer between the other two threads with the light gray thread above and the dark gray thread below. Since each layer is a torus knot, we see that the original link is *tori*-Brunnian in the sense that when any one thread is cut and removed, the remaining threads are tori-splittable into three torus knots.

5. Future Considerations

There are a variety of paths for future consideration. First, one could embed successively smaller Borromean Rings into the center region of a Borromean Ring. The result might not have a fractal look, but perhaps the center will provide an illusion of disappearing off to infinity. Second, a reviewer noted that hybrid objects that combine the first two procedures could be explored. In particular, the final image from Section could be embedded into the eight triangular regions in Figure 3.1. Third, it would be interesting to physically construct the spherical and the torus weavings. Notice that the mathematical challenge for the threads to separate into layers when any one thread is removed seems at first a very bad idea for fabric. Isn't Brunnian weaving opposite to the concept of "rip-stop"? Sometimes "yes". See [1] for an extreme case of *Brunnian Clothes* which rapidly unravel! And sometimes "no". When a thread breaks in fabric, one does not just remove it! In this paper, if a thread breaks and is not removed, the result might not be much different than from regular fabric. A broken thread on the Tori-Brunnian weaving is equivalent to a broken thread on regular fabric since locally they are equivalent. Returning to the idea of physically constructing these weavings, one notices that this ability to separate into layers provides a mechanism for the construction. The threads could be placed into position, on either a sphere or a torus, one at a time. Then, due to the Brunnian Property, only the last thread needs to be woven through all of the previous threads! The ability to create fabric that is not planar could have practical applications. Most likely it would be more useful to be able to weave a hemisphere rather than a sphere. Weavings with a boundary add an extra level of complication that was intentionally avoided in this paper. If you are interested in exploring these weaving, contact the author for the *Mathematica* code.

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