Bridges 2010: Mathematics, Music, Art, Architecture, Culture

# Gauss, Bolyai, Lobachevsky – in General Education?

## (Hyperbolic Geometry as Part of the Mathematics Curriculum)

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## Abstract

We describe an educational method to teach hyperbolic geometry at the upper elementary and secondary level, mostly with fruits and construction materials; and we make some remarks about the connection of hyperbolic geometry and poetry that explains the significance of such a method in mathematics education.

> It is not because of its complexity that I blame such an axiomatic system.
> It is the way in which it is offered to the students…
> Geometrical axiomatics cannot be meaningful as a teaching subject
> unless the student is allowed to perform these activities himself.
>
> (Hans Freudenthal)

Yes, the basics of the new geometry of Gauss, Bolyai and Lobachevsky should be part of general education. It has changed our concept of geometry, of mathematics, of the natural sciences, and of the world. After almost two centuries, it is simply a must for new generations to understand the basics of other geometries beyond the Euclidean one if they want to get any idea about modern science.

The only problem with such a statement is that almost any branch of mathematics makes similar claims. Half a century ago, the New Math Movement aimed at teaching teenagers the basics of abstract algebra which was just as important as geometry. Likewise, set theory, probability or cryptography also call for their places under the sun of education.

However, I think that importance is a necessary but not sufficient condition in this field. Forty years ago, Freudenthal was reluctant to include axioms of geometry into the curriculum of twelve-year-olds, not because the topic was not important enough, but because the axioms were set up by the teacher, not by the students. In other words, interest and independent activity of students are just as vital in the learning process as the scientific significance of the given theory.

The teacher must face an even greater problem in this case. Teaching about different geometries means the change of the paradigm of teaching itself. The greatest achievement of Gauss, Bolyai and Lobachevsky was, not the discovery of hyperbolic geometry, but the discovery of the existence of different geometries at the same time. Their discovery reminded me of the words of Galileo who did not claim that the heliocentric system was the only possible one. He only stated that the movement of the planets was easier to study by using the heliocentric model rather than the geocentric one. As he wrote in '*Dialogo*': 'It is easier to consider the pavement of the marketplace to be fixed rather than the children playing on the pavement.'

Still, the practising teacher often looks at 'Geometry' as an unassailable fortress of theorems that had been built by infallible luminaries of ancient times, as distant as possible. The teacher has the false conviction that the authority of ancient sages releases him from any responsibility for his teachings: 'This statement is theirs, not mine!'

**Figure 1.**

Fig. 1 shows a teacher performing experiments among her students. It is probably the first time in her life to study spherical geometry on an orange, with toothpicks and rubber bands. She may vaguely remember some lengthy trigonometric equations with many sines and cosines, but she has no other learned advantage over the thirteen-year-olds around her. Still, she is neither embarrassed nor pretending Know-All, but bravely takes up the role of the partner of her students in this research.

Alexandrov [1] wrote about hyperbolic geometry: 'Lobachevskian geometry can hardly be included in secondary school curricula, but it seems essential to give pupils an idea of it, to show the greatness of the human spirit, capable of creating unimaginable concepts and theories which in the course of time proved to be comprehensive and fruitful.' I disagree with the first part of this quote, but fully agree with the conclusion.

Following is a rough outline of a method that I call comparative geometry. The goal is to teach simultaneously Euclidean geometry via the model of a flat sheet of paper; spherical geometry on the surface of a fruit or a plastic sphere; and hyperbolic geometry on the surface of a plastic hemisphere or half an apple or onion. We only need the surfaces themselves to understand the basic concepts. To go beyond the basics, we can make use of planar and spherical construction materials.

So the essence of the method is to insert an intermediate step, a kind of spherical geometry, between Euclidean plane geometry and hyperbolic geometry. The spherical shape is familiar and friendly for any age group. Indeed, spherical or spheroidal forms are much more frequent in Nature than their planar counterparts. On the other hand, spherical geometry is far enough from the geometry of the plane to demonstrate the existence of a different world of geometry. If the student made the first step from Euclid towards the acceptance of spherical geometry, it will be much easier to make a further step from spherical to hyperbolic geometry.

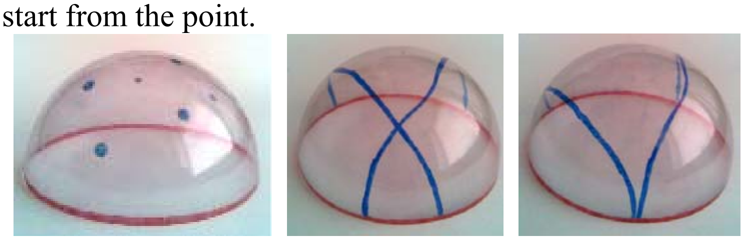
Luckily, a Poincaré model of hyperbolic geometry is built on the hemisphere. Experiences in spherical geometry come in handy when studying features of the hyperbolic hemisphere. Actually, it was the Poincaré model that helped me understand hyperbolic geometry. Before knowing this model, I was deeply frustrated by most of the books on the topic. I found that the word ‘obviously’ in these books would introduce a statement that was anything but obvious for me. The hemispherical model, however, seems to be viable not only for teachers-to-be (many of whom being frustrated in the same way with hyperbolic geometry as I was), but also for teenagers.

Naturally, there are many excellent software materials that can be complemented with comparative geometry, as for example, the program of Szilassi [5] or The Geometer’s Sketchpad [6] about hyperbolic geometry, or Cinderella [2] on comparative geometry.

So what are the surfaces on which we work out our geometries?

**Figures 2, 3, 4**

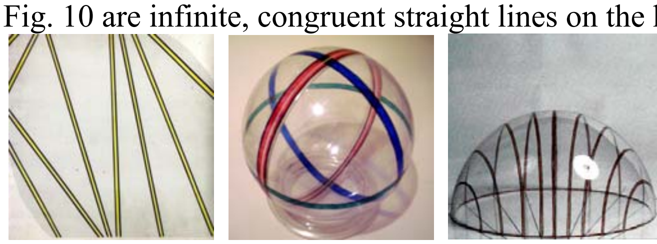
A sheet of paper that represents the infinite plane [Figure 2]; a plastic sphere whose surface represents the spherical surface [Figure 3]; and the open hemisphere whose borderline does not belong to our model represents the hyperbolic surface [Figure 4]. We choose the point as the basic element of our geometry in any of these three geometries [Figure 5 shows points on the hyperbolic surface]. We have but one reason for this choice, namely, we have been taught in school geometry to start from the point.



**Figures 5, 6, 7**

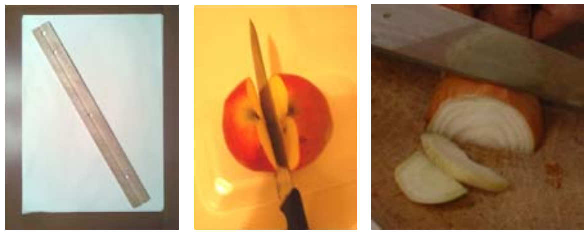
The two curved hyperbolic lines on Fig. 6 intersect, but on Fig. 7 they do not, because there are no points of the hyperbolic surface on the equator of the hemisphere.

We take for the simplest line the straight line on the plane [Figure 8]; the great circle on the sphere [Figure 9]; and the vertical semicircle on the hyperbolic hemisphere, that is, a spherical semicircle which is perpendicular to the omitted equator of the hemispherical model. All the vertical semicircles on Fig. 10 are infinite, congruent straight lines on the hyperbolic surface.



**Figures 8, 9, 10**

Other models of straight lines or segments: a straightedge on a sheet of paper; the cutting lines on the peel of a perfectly round apple that is cut into two halves; and the cutting line on the outer brown peel of a perfectly hemispherical onion sliced in the usual way on a slice board [Figures 11, 12, 13]:

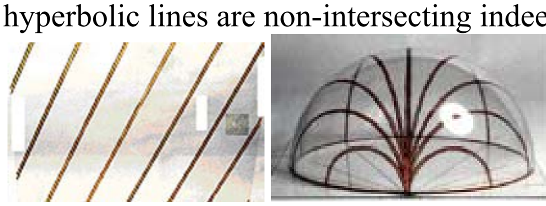


**Figures 11, 12, 13**

There are many ways of defining certain families of straight lines in the three geometries. One example of such definition is the concept of pencils of straight lines. On Figs. 14, 15, 16 there are three pencils each of which consists of all the straight lines passing through a fixed point of the surface. On the sphere of Fig. 15 these spherical straight lines pass through the opposite point as well. On the hyperbolic hemisphere of Fig. 16 we get funny spider-like lines that do not appear to be straight; but they are all hyperbolic straight lines.

**Figures 14, 15, 16**

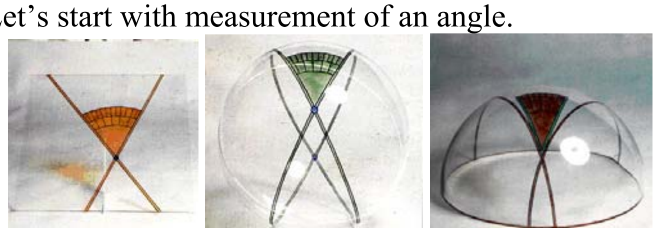
Other types of pencils consist of parallel straight lines. Fig. 17 shows a parallel pencil on the plane. No such pencil exists on the sphere, since there are no parallel straight lines on the sphere. Fig. 18 shows a pencil of parallel straight lines on the hyperbolic surface. Apparently, the lines on Fig. 18 meet in a point; but remember that the points of the equator do not belong to the hyperbolic model, so these hyperbolic lines are non-intersecting indeed.



**Figures 17, 18**

It is interesting to mention that many important results in geometry came from the intention to unify these two types of pencils (intersecting and parallel) into one single case; in other words, to consider parallel lines as lines meeting at 'a point at infinity'.

How to measure the distance between points and the angle between straight lines in the three different geometries? Let's start with measurement of an angle.

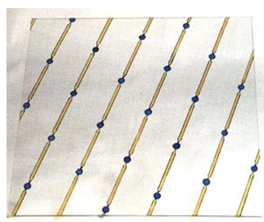
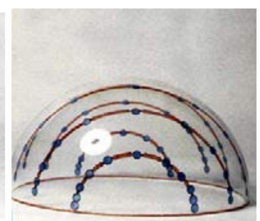


**Figures 19, 20, 21**

We measure the angle between two straight lines on the plane and on the sphere in the usual way. So the planar angle on Fig. 19 is 60°; and the spherical angle on Fig. 20 is 55°. In order to measure the hyperbolic angle on the hemispherical model of Fig. 21, we look for a moment at the figure as if the lines were not hyperbolic, but spherical lines. We draw two spherical tangents to the two semicircles at their point of intersection, and measure the spherical angle of the two tangents. Now we change our perspective again, and accept this measure as the measure of the two hyperbolic straight lines. So the hyperbolic angle on Fig. 21 is 40°. Here equal units of angle seem to be equal on each surface – in contrast with measuring hyperbolic distance, as you will see. We can believe our eyes when we measure angles on the three surfaces. However, another type of oddity comes up when we try to measure angle of hyperbolic straight lines.

On Fig 22 we see three types of pairs of hyperbolic straight lines. One pair has a point of intersection on the hyperbolic hemisphere, so we can measure their hyperbolic angle which is about 70° here. Another pair meets at the equator which has no point on the hyperbolic surface, so the two lines can be called parallel. In this case the two spherical tangents coincide, so their hyperbolic angle is 0°. The two straight lines of the third type (which look like two spherical semicircles) create no angle region to measure!

Measuring distance of points is easy on plane and sphere, but much more tricky on the hemisphere.

**Figures 22, 23, 24, 25**

Both on the plane and on the sphere equal units of measuring distance seem to be equal, but this is not the case on the hyperbolic surface. The equator does not belong to our model (we can imagine it as the horizon being infinitely far away from us), so its points can never be reached. So when you walk along a hyperbolic straight line towards the equator with equal units of measuring hyperbolic distance, your steps appear as becoming shorter and shorter for the external observer.

We can say, for example, that as we are approaching the omitted equator along a hyperbolic straight line, we always take half of the spherical distance on the line to the equator for the next hyperbolic unit. This means that our steps will become shorter and shorter for an outer observer, and we never reach the equator. The problem with this solution is that this measurement of distance does not fully harmonize with the chosen method of measuring angle. For example, it will not be true in a hyperbolic triangle that equal sides subtend to equal angles (the Isosceles Triangle Theorem). So we apply another way of counting equal units of hyperbolic distance. This counting is based on cross ratio – another concept which is not too difficult but a bit lengthy to explain here. The result is that, for example, the length of your steps would be, not 45, 22.5, 11.25, 5.625,… (always half of the previous one), but 30, 24, 16, ≈9.4,… Here we have infinitely many units again on each straight line.

All the lines of equal latitude (in the spherical sense of the term) are infinite hyperbolic straight lines, congruent with each other. The longitude-like lines show the scales of distance along the straight lines. Every latitude-like line on the picture has fourteen equal hyperbolic units of distance on it. The remaining, hardly visible parts between the lowest lines of measure and the equator are infinitely long hyperbolic rays. So those hardly visible parts contain incomparably more units than the visible parts.

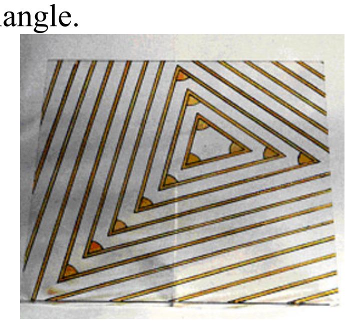
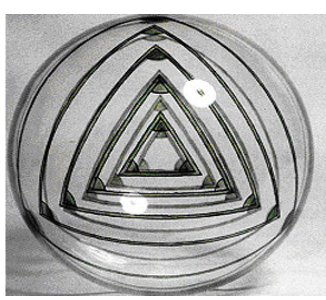
This construction can be used to design a hyperbolic ruler with a scale of distance on it as seen on Fig. 27. Here the ruler shows 10 units (five units of lighter colour and five units of darker shade), and the thin, non-coloured parts above the equator contain the infinitely long rays. The picture on Fig. 28 shows how to measure the hyperbolic distance of two points on the hemisphere. We draw the one possible hyperbolic straight line through the two points. Now we change our perspective from hyperbolic to spherical, and consider this segment of a hyperbolic straight line as if it were an arc of a spherical circle. We construct the centre of this spherical circle on the equator, and connect this centre with the endpoints of the arc. Then we fit the centre of our hyperbolic ruler to the centre of the circle, and measure the distance of the two points in hyperbolic units.




**Figures 26, 27, 28, 29**

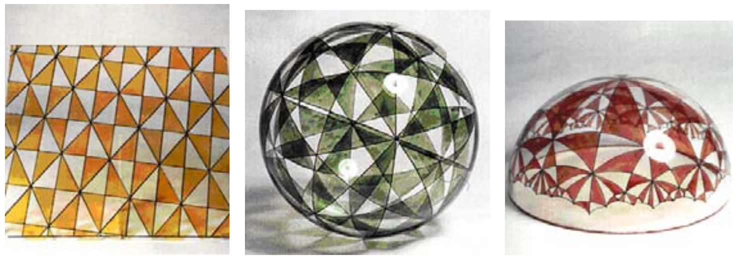
The hyperbolic distance of the two points on Fig. 28 is approximately 3 units – two full units plus a bit on the left side and a bit on the right side. Amazingly, if we make use of these methods and tools of measuring angle and distance, then the hyperbolic triangle on Fig. 29 is a regular triangle with three equal angles and three equal sides! The equality of the angles are clearly visible on the picture (they are about 30° each), but the equality of the sides can only be checked with the hyperbolic ruler. All the sides are about 5 units of hyperbolic distance.

What is the sum of interior angles in a triangle? The easiest way to check this sum is to take regular triangles, because we only have to consider one angle of each triangle (the other two are of the same measure). As you see in Fig. 30, the sum of angles in these triangles is always 60°+60°+60°=180° on the plane. On the sphere of Fig. 31, they start from 60°+60°+60°=180° because very small spherical triangles are very similar to planar triangles; but they keep on growing to the other extreme where all the three vertices lie on a great circle, and each angle will be 180°, so the sum will grow to 3·180°=540. On the hyperbolic hemisphere of Fig. 32, the sum starts from 60°+60°+60°=180°, but instead of growing, it diminishes to 0°+0°+0°=0° in the biggest triangle.


**Figures 30, 31, 32**

An interesting consequence is that on the sphere and on the hemisphere the change of length of the sides always goes together with changing the angles – in sharp contrast with the plane. In other words, there are no similar non-congruent triangles in spherical geometry and in hyperbolic geometry.



**Figures 33, 34, 35**

Figs. 33, 34, 35 show three tilings or tessellations or mosaics in the three geometries. Each of them consists of congruent triangles. This seems trivial on Figs. 33, 34, but far from trivial on Fig. 35. However, a closer look to the triangles of Fig. 35 shows that they have the same angles, 90°, 45°, and 30° in each triangle. So they are all congruent!

The famous 'Circle limit' drawings of Escher can be interpreted as if on a hyperbolic hemisphere, consisting of, not similar, but congruent motifs. Escher's drawings lead to another possibility of illustrating the hyperbolic surface, namely the Poincaré planar disc. (In Fig. 10, the hardly visible black segments under the hemisphere show straight lines of the Poincaré disc.) There are many other possibilities. One of the most interesting and appealing models of limited parts of the hyperbolic surface have been worked out by Daina Taimina in her 'Crocheted hyperbolic surfaces'[Figures 36, 37, 38].:

**Figures 36, 37, 38**

Apart from visual arts, there are bridges between hyperbolic geometry and literature as well.

Walt Whitman wrote around 1850, a few decades after the discovery of hyperbolic geometry, and a few years before its breakthrough among mathematicians: " The messages of great poets to each man and woman are, Come to us on equal terms, Only then can you understand us, We are no better than you, What we enclose you enclose, What we enjoy, you may enjoy. Did you suppose there could be only one Supreme? We affirm there can be unnumbered Supremes, and that one does not countervail another any more than one eyesight countervails another…".

Replace the word 'poets' with 'scientists' and read it again. This was exactly the spirit that led Gauss, Bolyai and Lobachevsky to a new world of geometry, and the spirit that led Whitman to a new world of poetry as well. It was the spirit that shocked the mathematical community, and the literary community of the nineteenth century. It is exactly the spirit that shocks many educators of our days. The reason is that this change means more than the change or extension of the material to teach and learn. This is a change of paradigm, from a fixed 'Supreme', a fixed system to teach to a menu of systems that the teacher offers to her students, and leaves to their decision which system they choose to use in a given problem solving situation. I think this is the key issue of the renewal of mathematics education.

At the beginning of the twentieth century, when Bolyai's work became widely known in Hungary, one of the best poets of the period, Mihály Babits wrote a poem titled 'Bolyai' (translated into many languages, the English translation made by Paul Sohar). Babits probably did not study hyperbolic geometry, yet he succeeded to express what was the pith of Bolyai's work even for the non-mathematician:

*With new natural laws, past the narrow sky,*
*I opened up a new infinity past the thinkable;*
*no king in history has conquered more than I*

*by stealing the secret treasures of the impossible.*
*Listen Euclid, your laws command you not to plod*
*beyond your prison; I just laugh at you with God.*

Let me finish off with some words about another type of bridges. Several years ago, I proposed to erect a memorial plaque which would commemorate Bolyai together with Gauss and Lobachevsky in Olomouc, where Bolyai served as an army officer. By the cooperation of the Czech Military Academy and the Hungarian Military Academy the idea turned into reality in 2004, but only with the name of Bolyai on the plaque in Czech and Hungarian languages.

But I still have my original dream. It would be wonderful to erect a common memorial to the three great spirits who represent three nations of Europe. Such memorial would serve as a bridge between nations, high over priority quarrels and petty fights. It would commemorate three great creators of one of the greatest achievements of mankind: GEOMETRY.

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[9] Figs. 36, 37, 38: Daina Taimina's crocheted models: plus.maths.org; sweetiepiepress.blogspot.com; scienceblogs.com
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