Imaginary Cubes
— Objects with Three Square Projection Images —

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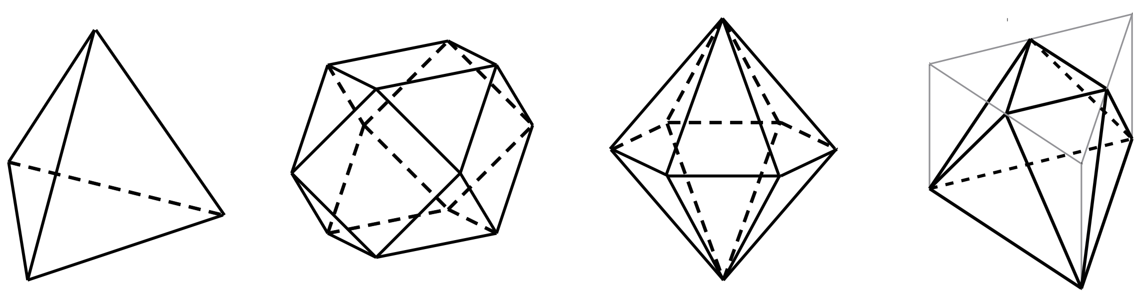
Abstract

An imaginary cube is a three dimensional object which has square projections in three orthogonal directions just as a cube has. In this paper, we study imaginary cubes and present sculptures based on imaginary cubes. We show that there are 16 kinds of minimal convex imaginary cubes. The list of 16 representative minimal convex imaginary cubes contains, as well as a regular tetrahedron and a cuboctahedron, a hexagonal bipyramid imaginary cube, which is a double imaginary cube, and a triangular antiprism imaginary cube, which plays an important role in the construction of the sculptures. Then, we present two Imaginary Cube Sculptures. Each of them is composed of all the 16 representative minimal convex imaginary cubes and forms an imaginary cube as a whole. Thus, they reveal uniform overall structures though they are composed of different shapes.

1 Introduction.

Imagine a three dimensional object which has square projections in three orthogonal directions. A cube has this property, but it is not the only answer and there are plenty of examples like a regular tetrahedron and a cuboctahedron (Figure 1 (a) and (b), see also No. 13 and No. 1 of Table 1). A regular octahedron (Figure 3(b)) also has this property and the intersection of the three right square prisms defined by its square projections is a rhombic dodecahedron, which also has this property. However, the inclinations of the three squares are different from those of cubes, and let us exclude them by restricting our interest to the case that the edges of a square shadow image are parallel to the other two orthogonal directions just like a cube. From an object with this property, one can imagine a cube which has the same three square projections. Therefore, we call such an object an *imaginary cube* (*I-cube* in short). Figure 1 (c) and (d) are two important examples of polyhedral imaginary cubes, which are explained in Section 3.



(a) (b) (c) (d)

Figure 1 : *Examples of polyhedral imaginary cubes, (a) a regular tetrahedron, (b) a cuboctahedron, (c) a hexagonal bipyramid which has 12 isosceles triangle faces with the height* $3/2$ *of a base, (d) a triangular antiprismoid obtained by truncating the vertices of one base of a regular triangular prism whose height is* $\sqrt{6}/4$ *of an edge of a base.*

In this paper, we study imaginary cubes and present sculptures based on them. In the next section, we study minimal convex imaginary cubes and show that there are 16 kinds of minimal convex imaginary cubes modulo rotational congruence. In Section 3, we explain some of them in detail. In particular, we show some remarkable properties of a hexagonal bipyramid imaginary cube and a triangular antiprismoid imaginary cube presented in [2]. After that, we move to non-convex imaginary cubes. In Section 4, we show that each Latin square defines a cubic imaginary cube. In Section 5, we present two imaginary cube sculptures each of which is composed of all the 16 representative minimal convex imaginary cubes and forms an imaginary cube as a whole.

2 Minimal Convex Imaginary Cubes.

For a cube C, we say that an object is an *imaginary cube of C* (*I-cube of C* in short) if it produces the same three square projection images as C. We start with the problem of characterizing I-cubes of a given cube C. First, if there is an I-cube of C, then its convex hull is also an I-cube of C. Therefore, we only consider convex ones from which all the I-cubes of C are obtained by making some hollows which do not change the three square projection images. Next, if a convex I-cube of C is given, then any convex object which contains it and contained in C is also a convex I-cube of C. Therefore, we are particularly interested in minimal ones. In what follows, we fix a cube C and study the problem of characterizing minimal convex I-cubes of C.

It is immediate that a convex object is an imaginary cube of C if and only if it is in C and has intersections with all the 12 C-edges. Therefore, a minimal convex I-cube of C is the convex hull of its intersection with the C-edges, which is a polyhedron. Thus, our first observation is that a minimal convex I-cube of C is a polyhedron all of whose vertices are on C-edges and each C-edge contains at least one vertex. Next, suppose that two vertices are on the same C-edge. If one of them is not a C-vertex, then we can remove it to have a smaller I-cube. Therefore, our second observation is that if there are two vertices on one C-edge, they should be the two endpoints.

As the third observation, if the three adjacent C-vertices of a C-vertex v are vertices of a minimal convex I-cube, then v is not a vertex of the I-cube by minimality. On the other hand, if a set of C-vertices which does not contain a vertex and its three adjacent vertices is given, then we can form a minimal convex I-cube of C by selecting one non-endpoint from each of the C-edges whose endpoints are not in the set. Note that it is minimal because if we remove one vertex, then it no longer satisfies the first observation. Therefore, we can obtain all the minimal convex imaginary cubes of C in this way.

As the “Cube-vertices” column of Table 1 shows, there are 16 subsets of the set of C-vertices which satisfy this condition if we identify rotationally congruent ones. Since No.10(L) and No.10(R) form a pair of mirror images and all the other ones have mirror symmetry, we have 15 subsets if we also identify reflectively congruent ones. We define that two minimal convex I-cubes of a cube C are rotationally (or reflectively) equivalent if they have the same set of C-vertices modulo rotational (or reflective) congruence. There are 16 (or 15) equivalence classes of minimal convex I-cubes of a given cube modulo rotational (or reflective) equivalence. This equivalence is natural in that when two equivalent minimal convex I-cubes of C are looked at from the same C-surface, one can see the same kind of polygons connected in the same way by edges.

Now, we simply say that an object is a *minimal convex imaginary cube* if it is a minimal convex imaginary cube of some cube. Though some imaginary cubes are minimal convex imaginary cubes of two different cubes, such an imaginary cube is in the same equivalence class (No. 5) for both of the cubes, as we will explain in the next section. Therefore, we can say that there are 16 (or 15) equivalence classes of minimal convex imaginary cubes modulo rotational (or reflective) equivalence.

In Table 1, we list imaginary cubes obtained by taking middle points of cube-edges as non cube-vertices. This choice of the representative of each equivalence class is natural in that, for almost all of the imaginary cubes in this list, the rotation group (or the full symmetry group) of such an imaginary cube is equal to that

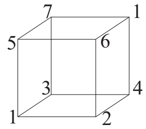
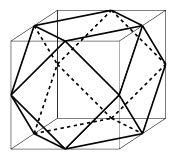
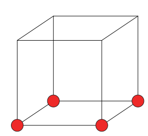
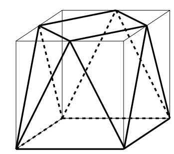
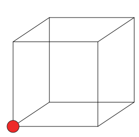
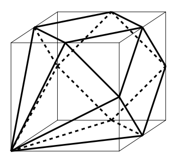
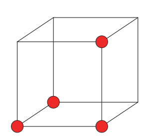
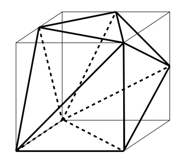
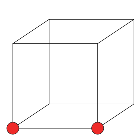
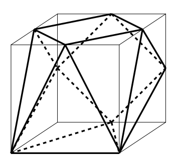
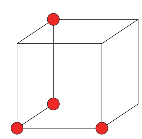
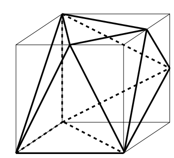
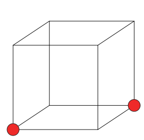
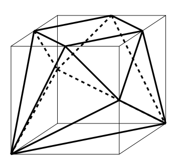
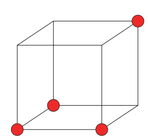
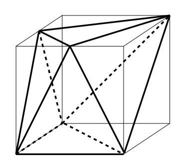
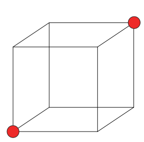
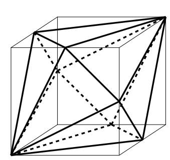
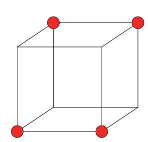
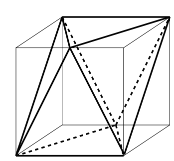
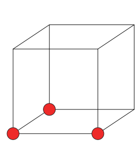
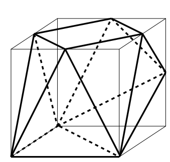
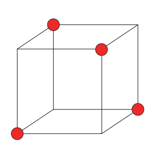
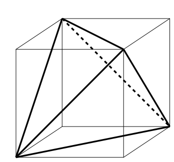
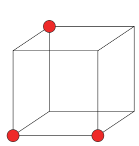
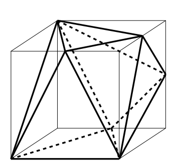
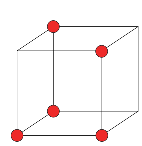
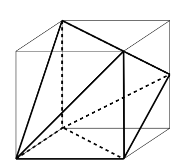
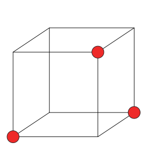
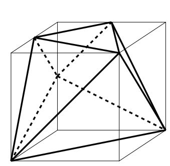
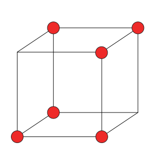
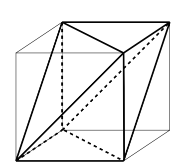
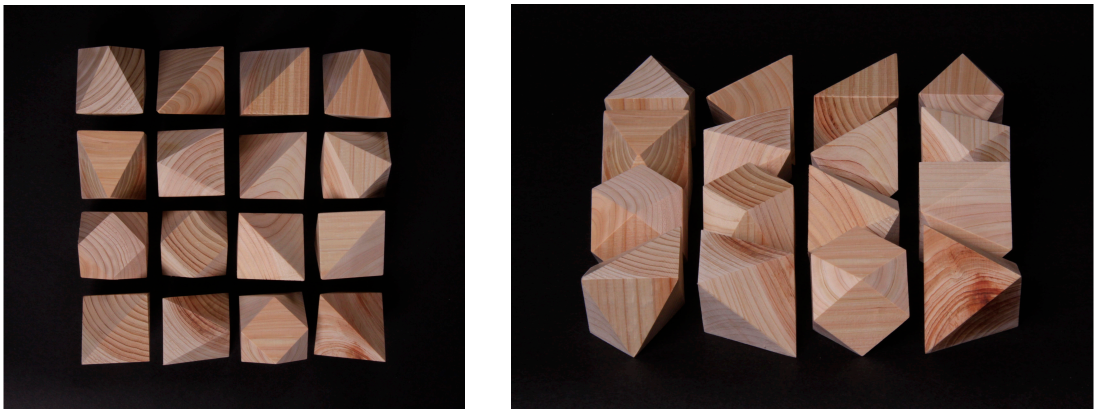
Number (#faces, #vertices)	Cube-vertices	Imaginary cube			
1 (14,12) Cubocta- hedron	 {}		9 (10,8) Quadric antiprismoid	 {1,2,3,4}	
2 (13,10)	 {1}		10(L) (10,7)	 {1,2,3,6}	
3 (12,9)	 {1,2}		10(R) (10,7)	 {1,2,3,7}	
4 (11,8)	 {1,4}		11 (8,6)	 {1,2,3,8}	
5 (12,8) Hexagonal bipyramid	 {1,8}		12 (8,6)	 {1,2,7,8}	
6 (11,8)	 {1,2,3}		13 (4,4) Regular tetrahedron	 {1,4,6,7}	
7 (10,7)	 {1,2,7}		14 (8.6)	 {1,2,3,6,7}	
8 (8,6) Triangular antiprismoid	 {1,4,6}		15 (8,6) Triangular antiprism	 {1,2,3,6,7,8}	

Table 1 : *The 16 (or 15) representatives of minimal convex imaginary cubes.*

of the corresponding figure of cube-vertices which is a cube with some vertices colored. The only exception is No.5, which has a bigger group as we will explain in the next section. Among these imaginary cubes, No.10(L) and No.10(R) are mirror images and all the other ones have mirror symmetry. Note that all of these representative imaginary cubes have rotational symmetry, even No.10(L) and No.10(R).

Normally, one cannot realize at a glance that a given polyhedron is an imaginary cube. The author has an exhibition at Kyoto University Museum displaying models of imaginary cubes and asking guests to put them in clear cubic boxes. It is not an easy task and it forms a good mathematical puzzle. Once an object is in a box, one can realize that it is an imaginary cube by looking at it from the surface directions of the box.



(a) (b)

Figure 2 : *(a) Wooden imaginary cubes. (Woodworks by Hiroshi Nakagawa.) (b) The same as (a) from a different inclination.*

One can also consider the same kind of question for regular octahedrons. That is, to characterize three dimensional objects which have square projections in three orthogonal directions just like a regular octahedron. For this case, a regular octahedron is the only minimal convex one, with a rhombic dodecahedron the maximal one.

3 Hexagonal Bipyramid and Triangular Antiprism Imaginary Cubes.

We explain more about some of the representative imaginary cubes in Table 1.

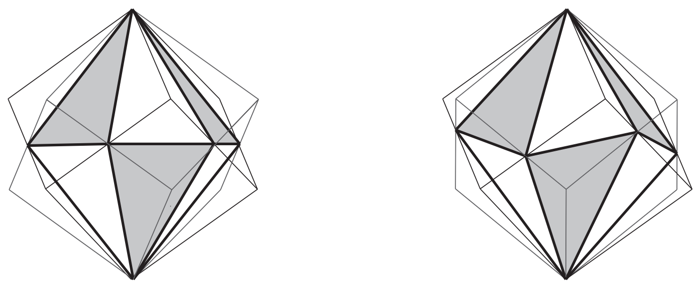
No. 1 (Figure 1(b)): Cuboctahedron. A cuboctahedron is an imaginary cube with no cube-vertices.

No.13 (Figure 1(a)): Regular tetrahedron. A regular tetrahedron is obtained by selecting as the vertices every other vertex of a cube. Therefore, it is an imaginary cube all of whose vertices are cube-vertices.

No 15: Triangular antiprism imaginary cube. A triangular antiprism with the sides right-angled isosceles triangles is an imaginary cube all of whose vertices are cube-vertices.

No. 5 (Figure 1(c)): Hexagonal bipyramid imaginary cube. This dodecahedron is composed of two copies of a regular hexagonal pyramid whose side faces are isosceles triangles with the height 3/2 of the base. It has a square appearance when it is looked at from each of the 12 faces. We call an imaginary cube a *double imaginary cube* if it has square projections in 6 directions which are divided into two sets of three orthogonal directions. This dodecahedron is a double imaginary cube and the two cubes of which it is an imaginary cube share a pair of opposite vertices and one cube is obtained by rotating the other one by 60 degrees. Note that this double imaginary cube is the intersection of the two cubes. It means that it is a maximal double imaginary cube as well as a minimal one and therefore it is the only convex double imaginary cube of the two cubes. The rotation group of this imaginary cube is the dihedral group D_6 of order 12, which is not a subgroup of the rotation group of a cube; half of the rotations of this object map one cube to the other cube. A fractal sculpture based on this polyhedron is presented in [2, 3].

More generally, consider the intersection of two different cubes which share a pair of opposite vertices. We have a dodecahedron which is a convex double imaginary cube of the two cubes as Figure 2(b) shows. One can easily show that all the convex double imaginary cubes are obtained in this way.

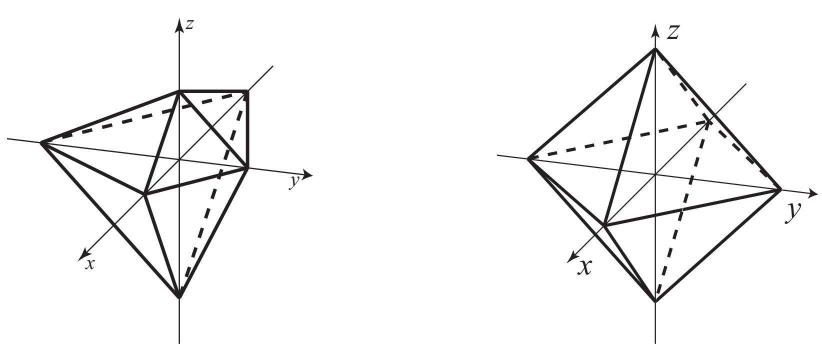


(a) (b)

Figure 3 : *Double imaginary cubes obtained as the intersection of two cubes. One cube is obtained by rotating the other one by (a): 60 degrees, (b): 42 degrees.*

No.8 (Figure 1(d)): Triangular antiprismoid imaginary cube. Consider the polyhedron constructed from a regular *n*-sided prism by truncating each vertex of one base at middle points of the edges of the base and the adjoining vertex on the opposite base. We call the prismatoid constructed in this way an *n*-antiprismoid. Figure 1(d) is a triangular antiprismoid constructed from a regular triangular prism with the height $\sqrt{6}/4$ of an edge of a base. On the other hand, the imaginary cube No.8 is an octahedron with two parallel regular triangular faces and the size of one of them is half of the other. It means that it is a triangular antiprismoid and through some calculation, one can see that Figure 1(d) and Table 1(8) are the same object.

Figure 4 shows yet another property of this octahedron. It has the property that the three diagonals connecting two opposite vertices intersect at one point and are orthogonal to each other, and the intersection point divides each of the diagonals by the ratio of 1:2. Therefore, with some coordinate system, the six vertices are on the three axes of coordinates and the distances from the origin to the three vertices on the negative sides are twice those on the positive sides. Note that if all the six vertices have the same distance from the origin, then it is a regular octahedron. This property is used in Section 5 for constructing Imaginary Cube Sculptures.



(a) (b)

Figure 4 : *(a) Triangular antiprismoid imaginary cube. (b) regular octahedron.*

Hexagonal bipyramid imaginary cubes and triangular antiprismoid imaginary cubes are studied in [2], and it is shown that they are closely related and they together form a tiling of the three-dimensional space.

4 Imaginary cubes induced by Latin squares

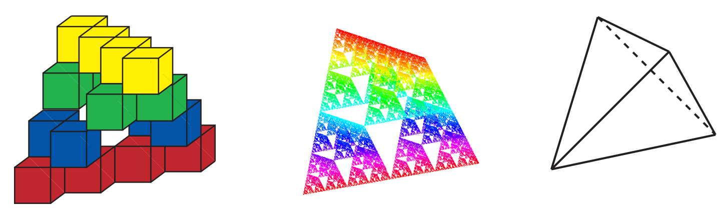
In this and the next section, we show a procedure to combine n^2 imaginary cubes into one imaginary cube following the structure of a Latin square of size $n \times n$ ($n = 2, 3, \ldots$). A Latin square [1] is an $n \times n$ table filled with numbers from 1 to n so that each number appears only once in each column and each row. When a Latin square is given, one can consider each number in a cell as the height and consider that it is specifying n^2 cubes out of $n \times n \times n$ lattice of cubes so that they are not overlapping from all the three surface directions. That is, a Latin square specifies an imaginary cube which is composed of n^2 cubes. We call such an imaginary cube a cubic imaginary cube of level n.

In a previous paper [2], the author considered such an imaginary cube as an iterated function system (IFS) composed of n^2 dilation functions with scale $1/n$ which map the cube surrounding the n^2 cubes to component cubes, and studied fractals with the fractal dimension two generated by such IFS's. He showed that a fractal obtained in this way is an imaginary cube, and thus a polyhedron defined as the convex hull of such a fractal is also an imaginary cube. When $n = 2$, there is only one cubic imaginary cube and the corresponding fractal is the Sierpinski tetrahedron, whose convex hull is a regular tetrahedron. When $n = 3$, there are two cubic imaginary cubes whose induced fractals have the hexagonal bipyramid imaginary cube and the triangular antiprismoid imaginary cube as convex hulls. When $n = 4$, there are 36 cubic imaginary cubes modulo rotational and reflective equivalences and there are only two connected ones, which are shown in Figure 5. One of them generates the Sierpinski tetrahedron as the fractal, as is the case for $n = 2$. The other one generates a fractal imaginary cube whose convex hull is a variant of a cuboctahedron, which belongs to the equivalence class No.1 in table 1. See [2] for the details of these fractals.

(1):

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

(1-a)

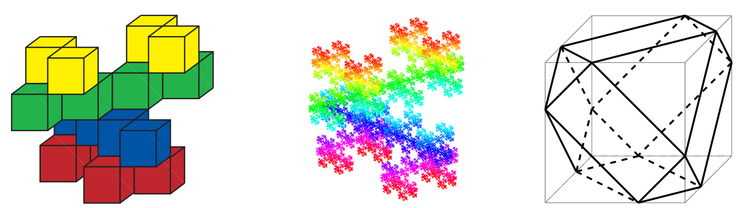


(1-b) (1-c) (1-d)

(2):

3	4	1	2
4	3	2	1
1	2	3	4
2	1	4	3

(2-a)

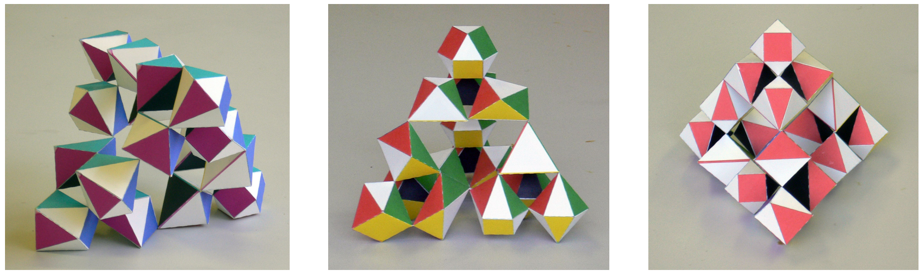


(2-b) (2-c) (2-d)

Figure 5 : *(a) Latin squares, (b) corresponding cubic imaginary cubes, (c) fractals generated by cubic imaginary cubes in (b), (d) convex hulls of the fractals in (c).*

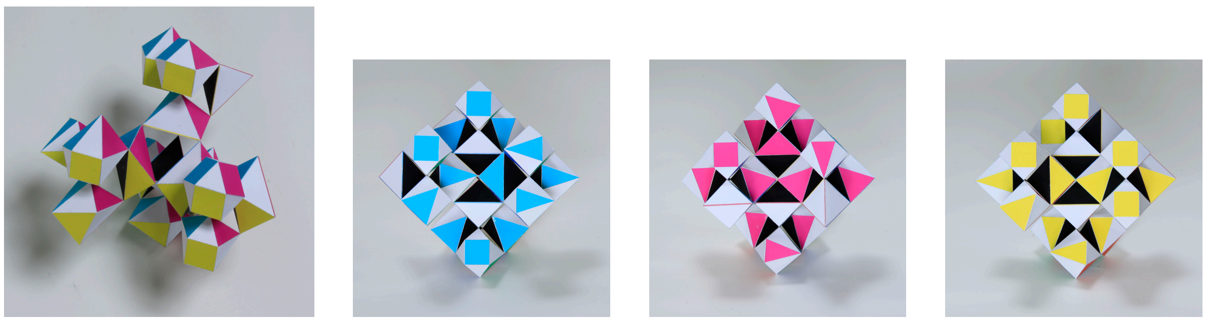
5 Combining Imaginary Cubes into one imaginary cube

Replacing the n^2 cubes of a cubic imaginary cube with n^2 imaginary cubes, one can form a new imaginary cube. As we have studied, there are 16 representative minimal convex imaginary cubes. The author combined them into two imaginary cubes according to the two cubic imaginary cubes (1-b) and (2-b) in Figure 5. Let us call them Imaginary Cube Sculptures with the Sierpinski Tetrahedron Layout (Sculpture #1 in short) and with the Cuboctahedron-like Layout (Sculpture #2), respectively. As we have noted these imaginary cubes are the only connected cubic imaginary cubes of level four, and they have the same symmetry as a regular



(a) (b) (c)

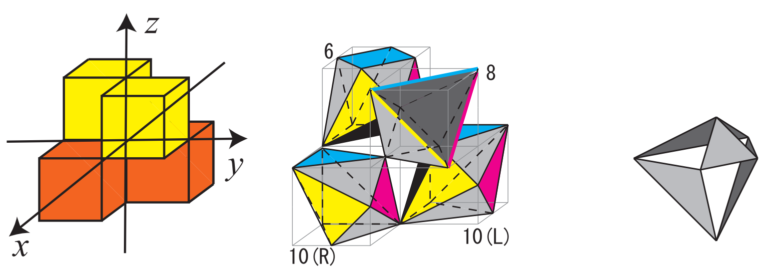
Figure 6 : *Imaginary Cube Sculpture – Sierpinski Tetrahedron Layout – (2010, Paper).*



(a) (c) (d) (b)

Figure 7 : *Imaginary Cube Sculpture – Cuboctahedron-like Layout – (2008, Paper).*

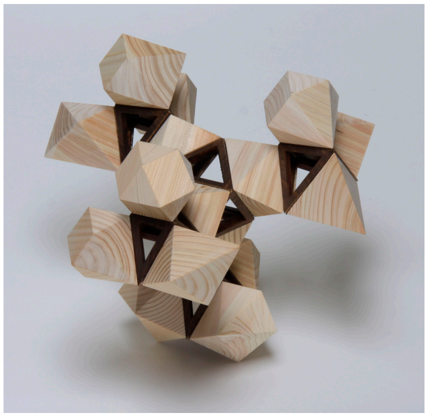
tetrahedron. He carefully arranged the allocation and orientation of the component imaginary cubes so that components in connected cubes share a vertex. Moreover, he arranged them so that the holes have the forms of a regular octahedron and a triangular antiprismoid, which we explain in detail.



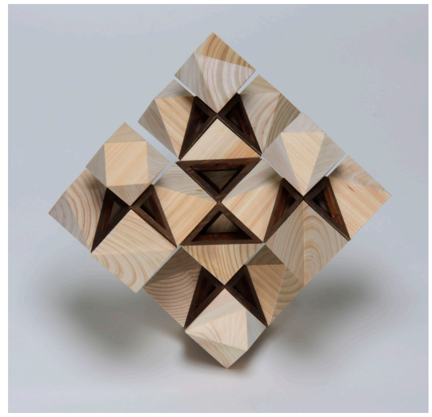
(a) (b) (c)

Figure 8 : *The triangular antiprismoid shape of the hole surrounded by imaginary cubes No. 6, 8, 10(L), and 10(R) in Sculpture #2. The hole at the center of (b) has the form (c).*

In the cubic imaginary cube Figure 5 (1-b) and (2-b), four cubes are connected in the form of Figure 8 (a) at four and five places, respectively. At each of them, adjoining imaginary cubes are not connected at the center of the four pieces. Therefore, there is a hole in the middle of them. If we have the axes of coordinates as in Figure 8 (a), around each hole, there are six points where two imaginary cubes meet and they are on the three axes of coordinates. The hole at the center of Sculpture #2 has the form of a regular octahedron, and all the other holes of the two sculptures have the forms of a triangular antiprismoid imaginary cube. Recall that the three lines connecting opposed vertices of a triangular antiprismoid imaginary cube are orthogonal to each other (Figure 4). In addition, Sculpture #2 can stand as in Figure 7(a) (and also Figure 9(a)) on



(a)



(b)

Figure 9 : *Imaginary Cube Sculpture #2 (2010, Wood, with Hiroshi Nakagawa).*

imaginary cubes 6, 10(L) 10(R) because they have cube-vertices at the bottom (Figure 8(b)).

They are colored so that those faces and edges with the same surface direction of the cube have the same color. Therefore, one can easily find the square appearances. When it is put as in Figure 7(a), yellow, magenta, and cyan come to the upper side and red, green and blue come to the lower side, with complementary color coming to the opposite side. Figure 7(b,c,d) are pictures from the upper three orthogonal directions.

The author assembled with Hiroshi Nakagawa a wooden version of Sculpture #2 (Figure 9). First, they formed dark-colored wooden frames of four triangular antiprismoids and one regular octahedron which consist only of the edges. Then, they glued the 16 imaginary cubes on faces of these frames.

Imaginary Cube Sculptures are composed of polyhedra with different shapes. However, they have uniform structures in that each of the components comes to be a square and the whole structure also comes to be a large square from each of the three orthogonal directions. Their overall shapes are also beautiful in that they roughly have the symmetry of a regular tetrahedron. Note also that the polyhedral shapes of the holes of Sculpture #2 make it possible to form a wooden sculpture though the components are only connected through vertices. More importantly, the component polyhedra of these sculptures are not arbitrary; they are the representatives of the set of minimal convex imaginary cubes.

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