

Strip Symmetry Groups of African Sona Designs

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Abstract

The sona drawings of the Chokwe people of Angola/Congo are continuous one-line drawings, generally made in the sand, that are often used to build highly symmetric one-line drawings. A large class of these drawings can be viewed as a form of “mirror curves,” visualizable as a curve bouncing through an arrangement of dots, bouncing off “mirrors” placed between some of the dots, and leaving a symmetric, closed trail behind. To investigate some of the possibilities for such Chokwe sona, we construct such sona for all 7 strip symmetry groups on all but the smallest rectangles. This allows us to use mathematical techniques to build classes of sona that, we argue, meet the artistic goals of the Chokwe people, and (we hope) would be viewed favorably by the historic artists of this culture.

1. Introduction & Terminology

The Chokwe people of Angola and Congo, and related peoples of the Chokwe-Lunda cultural groups, have a drawing tradition, done both in sand drawings and occasionally on more permanent objects, that has attracted a substantial amount of mathematical attention [e.g. 1, 4-10]. Although their sona (singular “lusona”) drawings arise in several different forms, one of the most common, and the most mathematical, consists of a grid of dots with a curve passing through the grid, “bouncing” off the external boundary of the grid, to form a single continuous curve. The “leopard with cubs” lusona shown in figure 1 is an example of such a drawing, with a few extra features added to the fundamental design to indicate the heads and tails of the mother leopard (horizontal) and her two cubs (vertical).

We say that a *sona grid* is a layout of uniformly spaced dots, which we will assume to be part of the integer lattice, with the property that the collection of dots is connected with respect to adjacencies in the unit lattice. We may view the lusona constructed from such a grid as being drawn by a curved line, drawn on the diagonals midway between the dots, and bouncing off the boundaries of the region specified by these dots. While this is a

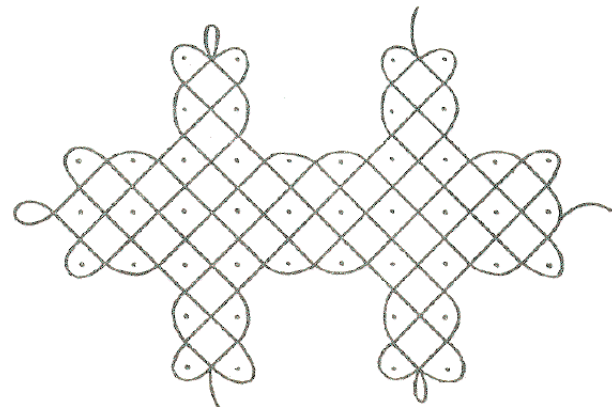


Figure 1: The “Leopard with Cubs” lusona, showing both how it might be drawn in the sand, and a more abstract representation of it.

useful metaphor, the curves often do not reach the sides of the bounding region to bounce off them like a strangely shaped pool table, but rather curve smoothly so as to just avoid hitting those edges, as shown in figure 1. See figure 5(c) for a lusona with bounces of both types.

Not all sona grids will give rise to acceptable sona drawings, however, because the Chokwe aesthetic requires (or strongly prefers) three criteria for such a drawing to be attractive to them. These criteria also give the sona tradition some of its interesting mathematics. The first requirement of the Chokwe aesthetic is that the line must be drawn as a single curve, while passing “smoothly” through each of the intersections—e.g., not turning a corner where two parts of the curve cross. For example, if we made the mother leopard of figure 1 one unit longer in both directions, then to construct the full lusona would require the 3 lines of figure 2. While this figure helps to demonstrate the way in which a Chokwe curve is drawn, the figure itself does not satisfy the Chokwe aesthetic because it is not monolinear. The Chokwe secondly require that each of the dots in a sona grid must be surrounded, and separated from the other dots, hence no subset of these three lines would make a valid lusona either. The third requirement of the Chokwe aesthetic is that the primary drawing (excluding any additional feet, tails, head, horns, etc. that are added afterwards) should be symmetric in some form, e.g. via rotation or reflection of the entire design, or translation or glide reflection of certain elements of the design.

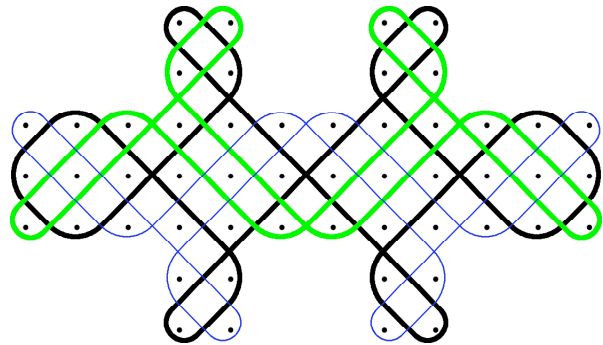


Figure 2: *The three curves required to draw the “Leopard with Cubs” if we had chosen to use a larger rectangle for the mother leopard.*

For many of the more interesting sona, the curve drawn by the Chokwe artist includes turns within the grid layout itself. This is demonstrated in figure 3 (left), a lusona of the Chokwe referred to as “ngonge tree”, and represents an insect which feeds on the inner bark of trees, leaving a pattern similar to this figure. One simple way to view this type of lusona is to imagine a series of walls inside a sona grid (figure 3, right), placed perpendicular to a line connecting two adjacent dots, and view the curve as “bouncing” off these walls. This sets these drawings within the mathematical framework of “mirror curves,” such as studied by Jablan [10] and others. We should note though that (1) the Chokwe artists do not draw these walls; and (2) while there is some evidence that the Chokwe artists may remember the drawings by remembering the placement of these un-drawn walls, that evidence is not definitive. The idea of using such walls, in a somewhat different drawing tradition, appears to have been first suggested by Bain [2], and adopting this idea to sona drawings was first suggested in Gerdes [8].

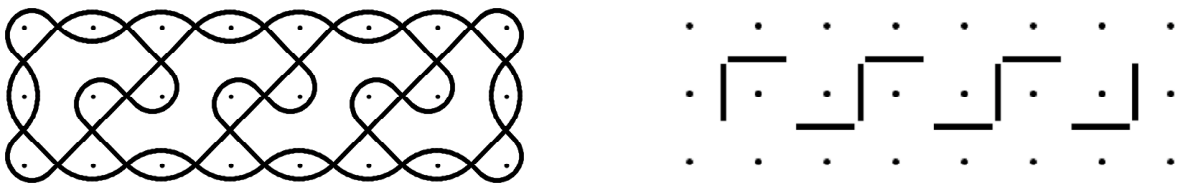


Figure 3: *An example of the internal “walls” or bumpers, on the right, that could be used to create the curves of the lusona on the left, built as a monolinear drawing.*

A *sona configuration* is a grid of sona dots together with a set (possibly empty) of internal walls. A sona configuration defines a unique lusona that can be drawn on that grid while respecting the bounce rules described above with respect to the boundary and the internal walls. We are thus interested primarily in sona configurations that require a single line to draw them and that are symmetric, in some way.

2. Other Types of Sona

Although the Chokwe people (a.k.a. Cokwe, Tshokwe) are the best known culture producing sona designs, many other cultures of the Central Bantu cultural area of Angola, Zambia, and the Congo produce similar designs. This includes the Nungo, Ruund, Lunda, Luimbi, Mbwela, Ngangela, Lucazi, Lwena, Mbunda, Nyengo, Ndembo, and Songo cultures. For simplicity, we refer to this larger group of cultures and artists by the name “Chokwe,” and do not distinguish between these groups. There seems to be no substantive study of the differences between the sona of these cultures. Because these groups have related traditions for these drawings, these drawings were collected historically only after the sona tradition began to decline, and since these cultures share many designs, investigators often use designs of one culture to infer likely historical patterns in the designs of other members of this larger cultural group.

What we described in section 1 is, actually, a special class of the drawings made by the Chokwe artists. Gerdes [8] refers to this class as “plaited-mat designs,” because of their similarity to the lines one creates with the strands of a diagonally plaited rectangular mat. In his analysis of 141 sona, he concludes that this is the largest class of sona designs. Other classes of sona exist that include elements of this form, but also include “wandering lines” connecting different portions of the designs. Figure 5(d) shows an example of such a design. Some examples of these “wandering line” sona use those additional curves for artistic elements of the design, while others use them to overcome the non-monolinearity of a lusona that the artist wants to construct. These other types of sona designs have also inspired substantial mathematical research, e.g. Gerdes [8], Demaine, et. al. [5], and Damian, et. al. [6]. The current paper focuses on plaited-mat designs.

Not all sona designs are monolinear, and not all are symmetric. Fontinha [7], in the largest single collection of sona, found that 80% of them were symmetric in some way. Gerdes’ analysis of 141 sona (see [8]) found that 61% of them were monolinear, but he gives convincing arguments that some of those that were not monolinear were so as a result of the informant trying to recall a design of an earlier artist, and recalling the details incorrectly. Gerdes then reconstructs the likely historical design by correcting these details. Thus the actual number of original designs that were monolinear is probably higher than this 61% value. In a few cases, the non-monolinear aspect of a design is a feature of that design, such as one where two independent lines represent the orbits of the moon and the sun, which interlace with each other, but never come together (Pearson [11], p. 93).

The conclusion of Gerdes [8], where chapter 3 is titled “Symmetry and Monolinearity as Cultural Values,” is that symmetry and monolinearity are both strongly desired features in the sona designs. Thus we set our challenge to meet their preferred values and try to design sona configurations that meet both of these goals. Further, since the largest class of sona designs are the plaited-mat designs, and since the artistic use of the “wandering lines” is clearly subjective, we also constrain ourselves to producing symmetric, monolinear, plaited-mat designs, i.e. designs which follow the rules of section 1.

3. Symmetric Sona and Related Work

A large number of the authentic Chokwe sona have some type of central symmetry, i.e. they have cyclic (C_n) or dihedral (D_n) symmetry groups, with C_4 and D_2 being particularly common. We consider this type of lusona in [3]. That paper classifies exactly when a grid of dots with symmetry group C_2 , C_4 , D_1 , or D_2 has an arrangement of walls that determines a lusona with the same symmetries (almost always, except when a parity argument shows it to be impossible). That paper is less successful in analyzing when grids with D_4 symmetry have a lusona with that symmetry, but that may reflect the reality that the Chokwe artists themselves are also somewhat unsuccessful at finding such sona.

Several Chokwe sona designs have regular 2-dimensional symmetric designs on the inside, with an exterior “frame,” which is also regular. Figure 4 shows an example, known as the “Lion’s Stomach.” In [4], we analyze a variety of classes of sona of this form that satisfy all of these possible “wallpaper” symmetry groups.

In this paper, we focus on the 7 strip symmetry groups and sona designs which demonstrate these symmetry groups. We begin by surveying some of the authentic sona that demonstrate such strip patterns, and then proceed to construct sona of all 7 strip symmetry groups for essentially all linear rectangular sona grids.

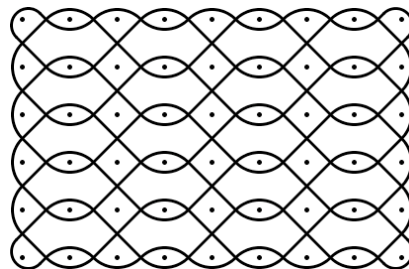


Figure 4: A monolinear sona whose interior is the wallpaper symmetry group pmm .

4. Chokwe Sona with Strip Symmetry Designs

Before we investigate the existence of sona with various strip symmetry groups, we should provide evidence that this is not imposing our own mathematical views onto the art forms produced by the Chokwe. Our goal is to produce sona that these cultures themselves would enjoy, and would feel belong to the type of art that they produced. To make this argument, we want to look at existing examples of sona designs and verify that the traditional artists produced sona designs with these various symmetry groups. A large majority of extant symmetric sona have central symmetry, e.g. 79% in one sample (see Chavey [3]). Nevertheless, there are several sona with linear symmetry. In analyzing those sona collected in Fontinha [7] and Kubik [9], we found the following distribution of linear strip symmetry groups among the plaited-rug sona and the more general sona:

Symmetry Group	p111	p112	p1a1	p1m1	pm11	pma2	pmm2
Plaited-Rug	1	1	0	3	1	2	23
Others	2	3	3	2	9	3	21

Table 1: The number of sona designs with the 7 different strip symmetry groups.

This table is surprisingly subjective. With such designs, it is not always clear whether to classify them as centrally symmetric, or as a strip pattern (i.e. a portion of an infinite strip pattern). For example, figure 5(d) as a whole has only bilateral symmetry, but the main portion of this design consists of a lusona with linear strip symmetry. We have tried to classify a design as having a strip pattern when it appears that the artist’s intent was to construct a linearly repeating design, possibly as a major portion of a larger design.

Clearly, the majority of these linear designs have $pmm2$ symmetry, i.e. designs with both horizontal and vertical reflections. Many of these designs are built from the basic rectangular sona grid with no interior walls. (Such a design is monolinear whenever the dimensions of the rectangle are relatively prime.) However, there are enough designs of the other types to convince us that the Chokwe artists enjoyed and appreciated these other types of symmetries as well. We include examples of sona from each of these symmetry classes. Specifically:

p111: The lusona of figure 5(e), [7, fig. 137] is part of the symbol of a particular cult. This lusona has only translation symmetry, although it appears on the diagonal—which is fairly unusual for this type of lusona. (The full symbol has additional elements to the upper right of this part.)

p112: Figure 3 shows a lusona whose symmetry group is $p112$ (rotations of $\frac{1}{2}$ only) [11, p. 162].

pm11: Figures 5(b) and 5(d) are non plaited-rug sona with $pm11$ symmetry (vertical reflections) [7, figs. 159 and 350].

p1m1: Figure 5(f) is a lusona with p1m1 symmetry (horizontal reflections), with the tails [7, fig. 218].

p1a1 and pma2: The “Leopard with 5 Cubs” lusona of figure 5(a) [7, fig. 191] has pma2 symmetry (glide reflections + vertical reflections) if we ignore the tails. Including the tails, it has p1a1 symmetry (glide reflections only).

pmm2: Many Chokwe designs are based around a $2 \times n$ or $3 \times n$ rectangle with no internal walls, such as figure 5(c) [7, fig. 169], based on a 2×5 rectangle, or the 3×10 rectangle at the core of figure 1.

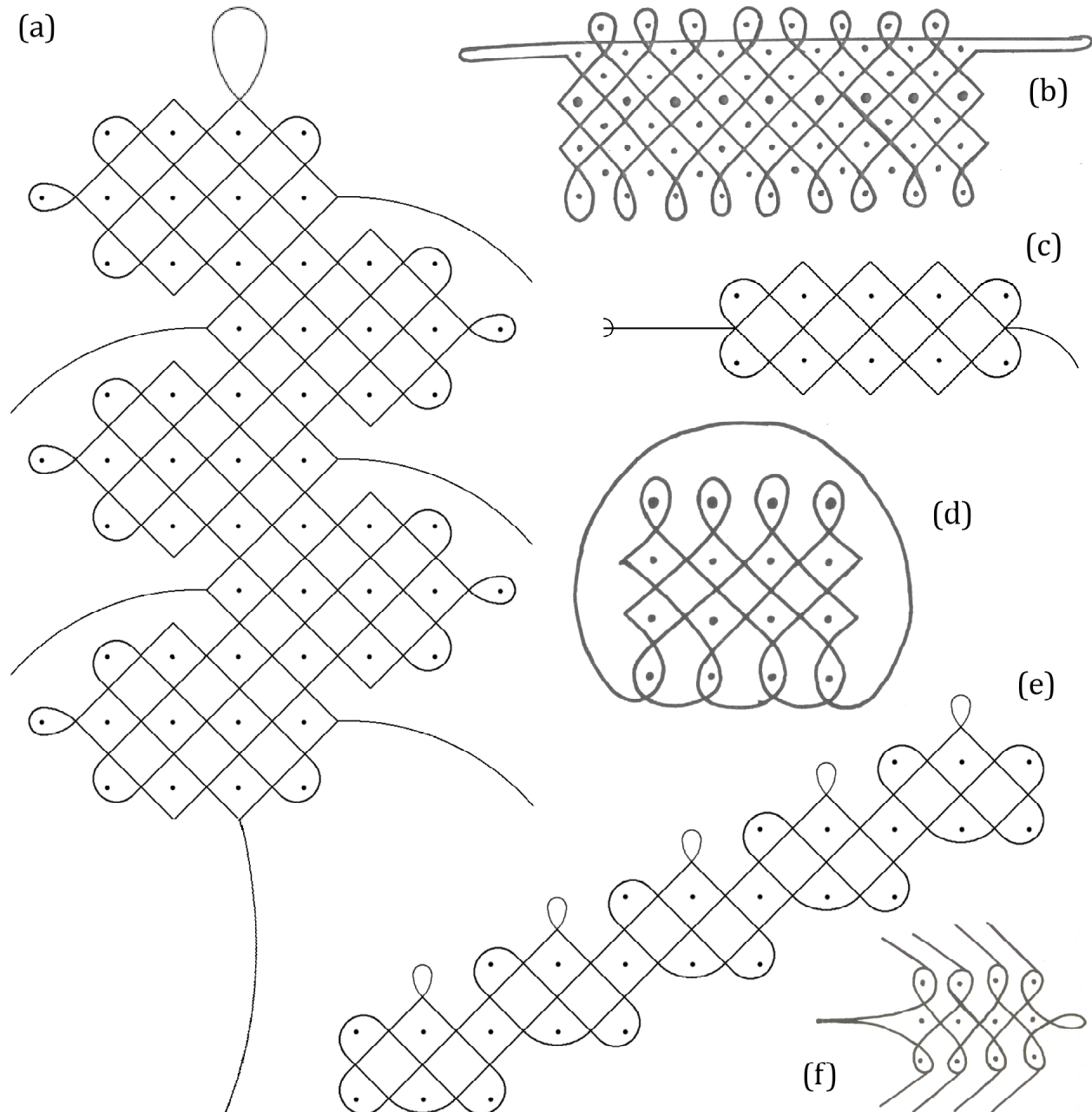


Figure 5: Examples of sona with linear symmetry. In clockwise order, from the left: (a) Leopard with 5 Cubs; (b) A sedan chair for carrying notable people; (c) A kind of rat; (d) A protected settlement; (e) The main portion of the mark of a cult; (f) A porcupine.

4. Existence of Linear Symmetric Sona

To demonstrate the existence of rectangular sona with various strip symmetry groups we assume that we are given an $n \times m$ rectangle, with $n \leq m$ (where generally n is significantly smaller than m). We would like to show how to add walls to construct a monolinear lusona with strip symmetry, where that symmetry might be broken at the shorter two edges, i.e. at the right and left ends of the strip. Specifically, we prove:

Theorem: Given an $n \times m$ rectangular grid of dots R_n , with $m > n \geq 5$, and $m \geq 12$ being the horizontal dimension, and given any of the seven strip symmetry groups S , there is a set of walls such that R_n with those walls generates a monolinear lusona which (excluding the left- and right-most columns) is a segment of an infinite strip pattern whose symmetry group is S , and which consists of at least two fundamental regions of that infinite strip.

Proof: We first show the existence of such sona with $pmm2$ symmetry, building up a lusona from a “spine” along the horizontal center, with symmetric elements above and below the spine. If n is odd, we begin with a $1 \times m$ spine, which is always monolinear. If n is even, we begin with a $2 \times m$ spine, which is monolinear if m is odd. If m is even, we add a single wall separating the two rows at the far left (see figure 6), which gives a monolinear lusona as our starting point. Since $n \geq 5$, the width of the remaining strip above the spine is at least 2, and hence can be expressed as the sum of some number of 2’s and 3’s. An easy induction argument shows that any number 7 or larger can be written as the sum of a sequence of 2’s and 3’s that use at least one 2 and one 3. Thus if $m \geq 14$, the length of the strip can be written as at least 2 periods of such a sequence (e.g. $14 = (2 + 2 + 3) + (2 + 2 + 3)$), and since $12 = (2 + 3) + (2 + 3) + 2$, and $13 = 3 + (2 + 3) + (2 + 3)$, those values of m can be written as approximately $2\frac{1}{2}$ periods of a pattern of 2’s and 3’s. (To prevent unintentional symmetries later, the periods should be written with all 2’s first, followed by the 3’s, as in these examples. The need to include both blocks of size 2 and of size 3 also prevents such unintentional symmetries.) The components of figure 7 can be added to the top or bottom of any monolinear lusona without changing the linearity of that lusona. Since these components have bilateral symmetry, adding them above a central spine, and their reflected versions below the spine, maintains the $pmm2$ symmetry of the strip. When we add these components to a spine, we use vertical sona walls to separate the components from each other and stack the components above and below each other to build up the desired rectangle while maintaining this monolinearity. Figure 8 shows the results of adding these components to a 1×18 spine to create an 11 high by 18 wide strip with $pmm2$ symmetry. We omit the sona walls, but use dashed lines to show the locations of the individual components on the left.

We can modify this process to create patterns with symmetry groups other than $pmm2$. For example, by replacing the components of figure 7 with those of figure 9, possibly reflected, and in different orientations in different places within the pattern (e.g. top or bottom, odd or even positions, etc.), we can convert any lusona built via the process of the previous paragraph into one having $pm11$, $plm1$, $pl12$, or $pl11$ symmetry.

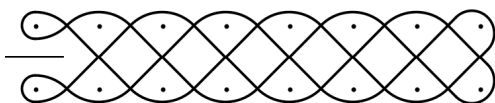


Figure 6: The center spine of a rectangular strip with even height and width.

	2 dots wide	3 dots wide
2 dots high		
3 dots high		

Figure 7: Component pieces of various sizes which, added to a spine, do not change the monolinearity of the sona.

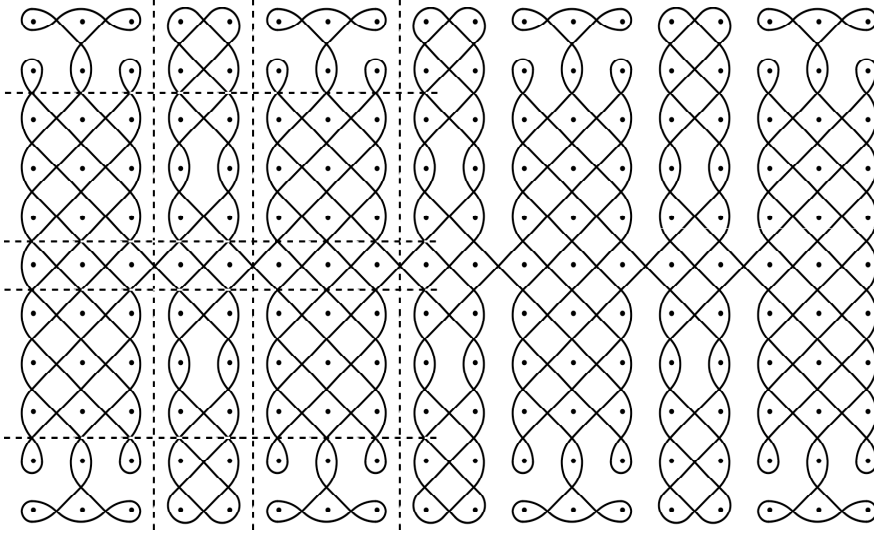


Figure 8: Building up a monolineal sona with $pmm2$ symmetry from the components of figure 7.

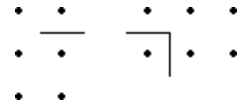


Figure 9: Asymmetric components used to build symmetric sona.

For $pma2$ and $plal$ symmetry, we need to embed glide reflections in the pattern. To do this, we must allow portions of the right-most and left-most columns to break with the main pattern, e.g. parts of these columns may be a single $1 \times k$ column of dots. To satisfy the theorem’s requirement that the pattern “consists of at least two fundamental regions of that infinite strip,” it is sufficient that we have one full pattern above the spine, and the full glide-reflected image of that pattern below the spine.

Any even number larger than 8 can be written as the sum of a sequence of 2’s and 3’s which includes some of each, and an even number of 3’s. Furthermore, any sequence of blocks of length 2 and 3 can be cycled in such a way that the center of the full block is either adjacent to a block of size 2, or splits a block of size 2, while maintaining the property that the blocks of length 3 are contiguous, although that continuity may require wrapping from one end to the other. Figure 10 shows examples for length 8, 10, and 12. By using these blocks, one glide reflected image of them, and allowing ourselves one extra dot on one end or the other (when the center line splits a block of size 2), we get blocks of dots with glide-reflection symmetry, such as in figure 11. Pattern imperfections here come from the finiteness of the design and the occasional “extra” dot on the right or left. Since this construction requires a length of 1.5 times the original block length, we can generate such sona for any length m that is a multiple of 3. For values of m where $(m \bmod 3) = 1$ or 2, we add single columns of dots to the right and/or left of this construction. Because of the placement of the blocks of size 2, single dots of the construction become full blocks, and at most one column on the right or left may disrupt the pattern. As with the previous construction, we then use blocks of width 2 or 3, building to the appropriate heights above and below the spine, and using symmetric blocks to force $pma2$ symmetry, or asymmetric blocks to force $plal$ symmetry. An example of this construction for a 5 by 19 rectangle with $plal$ symmetry is shown in figure 12. ■

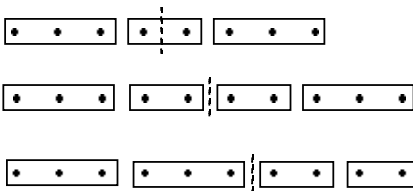


Figure 10: Blocks used to build strip designs with glide reflections. The dashed lines show the centers.

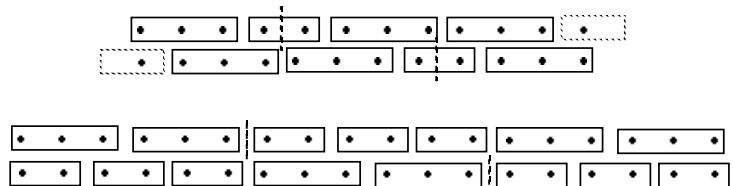


Figure 11: Glide-reflective patterns built from the 1st and 3rd block arrangement of figure 10.

Variations of these techniques, and artistic experiments with individual strips, can give a wide variety of other strip-symmetric sona. The techniques used here generate less variation than ones aimed at a particular shape or design, but are successful at generating such sona for almost any size strip, with any symmetry group. We feel that examples such as figures 8 and 12 demonstrate that these proof-generated constructions still generate sona that are aesthetically pleasing, fit within the cultural and artistic norms of the sona-drawing peoples, and generate sona that we hope the original artists would enjoy.

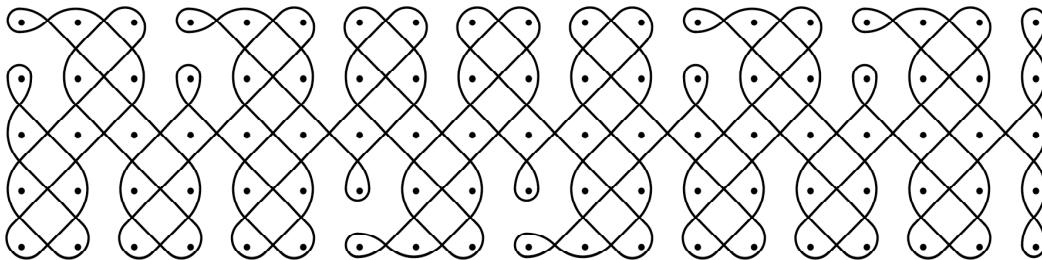


Figure 12: *The construction from the proof of a 5 x 19 rectangle with strip symmetry group $plal$.*

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