Notation For a Class of Paperfolded Models

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Abstract

We propose notation for a class of paperfolded models that uses a simple formula-like string to completely describe the folding process.

1 Introduction

In origami, or paperfolding, there are models that require complex fold steps. In order to describe the folding process in written form, over time an intricate system of diagramming symbols and conventions has been developed. Even so, there have been models that simply defy diagramming and whose authors (designers, or composers) prefer to teach the folding process only in person, or not at all, and instead provide only the *crease pattern* that shows the location (and sometimes sense) of all the folds actually present in the finished model. Of course, a crease pattern alone may not uniquely define the model, but in typical cases it allows for an educated guess at how, for example, the layers of paper should be arranged.

1.1 Pureland origami

So-called *pureland* origami models restrict the folds used in each step to just the basic *mountain* and *valley* folds. In a valley fold, a straight line from one edge of the model to another divides the (normally flat) model into two parts. The fold consists of a single crease along the straight line, and except for the rotation of one of the two parts formed by the crease around the crease, nothing else moves. Pureland origami has been popularized by John Smith through several booklets published by the British Origami Society. Due to the simplicity of individual steps, it is particularly well suited to the teaching of beginners. Since the nature of a simple mountain or valley fold is to rotate one part of the model around the fold crease by 180 degrees, pureland models almost always remain flat.

1.2 Tessellations

As the name suggests, paperfolded *tessellations* are formed by repeating a single pattern, or a small set of patterns, across the whole folded sheet. The folding of tessellations is usually considered to date back to the work of Shuzo Fujimoto [3] and Yoshihide Momotani [7] in the 1960s and 1970s in Japan, although there are known examples of folded tessellations as early as this in the West as well, notably that of Ron Resch. Most folded tessellations require higly complex steps, since the different instances of the pattern to be folded tend to interfere with each other and in fact, the easiest way to fold some tessellations is to carefully coax the precreased or scored sheet into forming all the creases at once. In contrast to such more typical folded tessellations, we study a class that can be folded by iteration of simple mountain and valley folds, that is, a class of *pureland tessellations*.

A *pleat* is formed by a pair of adjacent parallel folds, one a mountain and the other a valley fold.

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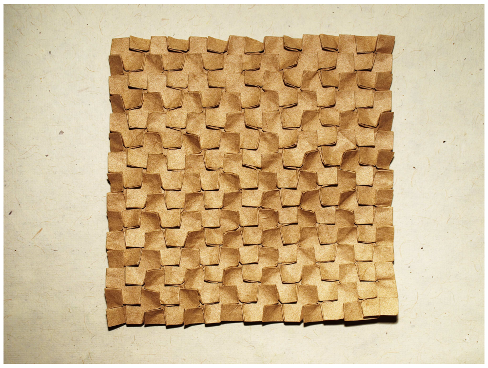
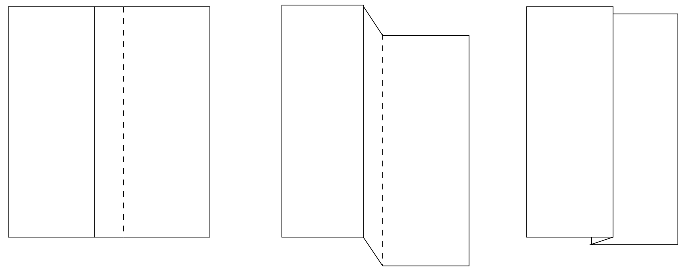


(a) A tessellation fold by T. Kawasaki



(b) Folding a pleat. Mountain fold (solid line) next to a valley fold (dashed line).

Figure 1: A tessellation that requires complex fold steps or a nonrigid collapse to fold; simple pleat fold.

2 Pleat tessellations

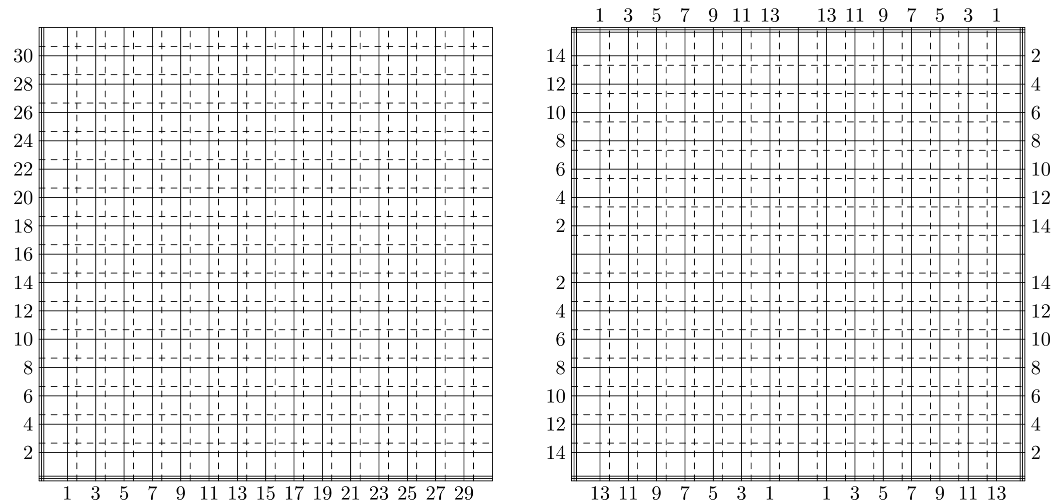
We consider pleat tessellation models formed by folding a sequence of pleats along grid lines parallel to the edges of the initial square, and possibly a few extra valley or mountain folds (the lock). There is no unfolding, so the model belongs to "pureland origami" and in theory lies flat. However, the photographs of models shown in these pages illustrate the difference between the theoretical model and the real life: paper has finite positive thickness and the models shown are obviously not flat. They *can* be forced into lying flat by careful application of force, but as soon as they are released, they spring back into a three-dimensional shape. This behavior is the result of a locking fold applied to (some) edges of the folded sheet. The lock consists of two parallel mountain folds close to the edge, and keeps the pleats folded tightly at the edge. The shape itself is determined by the arrangement of pleats. As in the case of the hyperbolic paraboloid [2], the three-dimensional shape is determined by the tension of the folded sheet, but in this case the thickness of the paper and the elastic properties of its folds play a very significant role. It is an interesting and apparently difficult problem to predict the exact shape formed by a pleat tessellation model. While there has been some work on computer simulations of the physical behavior of folded paper [1, 6, 4], none of it addresses the forces acting between adjacent layers of paper. We are currently studying a mathematical model that allows us to take this issue into consideration. The notation presented here is the (easy) first step in building such a simulation, since it allows a simple input specification to be translated into the folding process and an initial state for the simulation to be computed.

Pleat tessellations appear to have been folded first by Paul Jackson, and a photograph of the basic form, titled "Bulge", appears in his Encyclopedia of Origami & Papercraft Techniques [5]. The basic form has also been used by several origami designers to represent fish scales or snake skin. The more complex pleat tessellations shown here have been folded by the first author, starting in 2006.

3 Pleat tessellation notation

Since all pleats are parallel to the edges of the original square, each pleat is completely specified by four pieces of information: **(a)** *direction* (horizontal or vertical), **(b)** *location* (the coordinate of a point at which the mountain fold of the pleat intersects an edge of the square), **(c)** *sense* (positive or negative—the sign of the difference between the location of the pleat's mountain and valley folds), and **(d)** *width*.

Clearly, each pleat tessellation can be completely described by listing all the pleats to be folded in order. Thus to explain how to fold a pleat tessellation, we need no diagrams!



(a) The basic crease pattern with pleat order specified. The mountain folds very close to bottom and left edges are the optional locking folds

(b) Bottom and left edges: pleat order for flat-bottom bowl or pyramid (locking optional). Top and right edges: pleat order for simple bowl (with lock) or "four peaks" (without lock).

Figure 2: Crease pattern for models in Figures 3(a)–3(d). Each pleat is given a rank in each model. The pleats should be folded in the order specified by these ranks. Pleats assigned equal rank can be folded in any arbitrary order. Each number refers to two creases that form a single pleat. Figure 2(a) shows the pattern for the basic form in Figure 3(a). The bottom and left edges of Figure 2(b) show the sequence of pleats that leads to the pyramid of 3(c), while the top and right edges define the models in Figures 3(b) and 3(d).

We describe two ways to notate a pleat tessellation. The first is very general, and could be generalized to an arbitrary pureland model. There are several precedents for this, most notably the origami instruction language OIL [8]. However, thanks to the restrictions we impose on the creases in pleat tessellations, our task is simpler. In fact, in many cases the notation for a sequence of creases can be compressed, resulting in very short descriptions of some models that appear relatively complex. The second way of notating (typical) pleat tessellations is even more compact, because in addition to individual creases, it uses the fact that typical pleat tessellations are formed by folding one or more regular sequences of pleats, in ways that create symmetric patterns, and that there is only a small number of different ways these sequences are composed and interleaved.

We first describe the general notation for a single pleat, and then, after giving some examples of pleat tessellations defined by listing all their pleats, show some simple ways of compacting the description. Finally, we describe the compact notation.

4 Single pleat

Direction. Each pleat is by definition either horizontal or vertical, and so we use the letters *h* and *v*, respectively, to denote the direction.

Location. It is convenient to express distances and lengths in terms of the size (side length) of the original square—that is, a horizontal fold formed by placing two opposite edges of the square on top of each other and pressing the sheet flat would be at distance $1/2$ from the edge of the square. A pleat that uses this crease as its mountain fold would then be at location $1/2$. On the other hand, if pleats that we fold are all located on a fixed grid, then writing down just the numerators of their locations creates significantly less clutter. In such a case, though, the numbers are meaningful only if we know the grid size. For example, if we start with a 16×16 grid, the same horizontal crease through the center would have location 8, while in a 32×32 grid its location would be 16. Which of the two units is being used may be indicated, for example, by writing $u = 1/16$. However, since in the first case all the distances are less than 1, and in the other all the distances are at least 1, which unit is used can normally be inferred from the description.

To be consistent in this notation, we set up a coordinate system with origin at the bottom left corner of the square and direct the x and y axes in the standard way, towards the right and up.

Sense. If we fold a horizontal valley fold, and then some distance above it a horizontal mountain fold, we get a positive pleat. If the mountain fold is below the valley, the difference between the mountain and valley locations is negative, and so the pleat is negative as well.

Width. Again, this is expressed either in terms of the original square size, or in terms of the grid unit distance. In the notation, we will combine the information about the sense and the width of the pleat, by writing the width of a negative pleat as a negative number.

Pleat notation. A pleat formed by a horizontal mountain fold through the center, and a valley fold $1/16$ of the square width below it would be written as $(8/7)h$ (the mountain fold is at distance 8 from the bottom of the square, the pleat is horizontal, and the valley fold is (positive) 1 units below the mountain fold). If the pleat were negative (valley through the center, mountain a unit below), we would write $-(8/7)h$. The simple rule for decoding this notation is: the first number tells us the location of the top crease of the pleat, and the letter the direction. If the pleat is positive, then the mountain fold is above the valley, so the location of the mountain fold is given by the first number. If the pleat is negative, then the first number gives the location of the valley fold.

5 General pleat notation

The basic form. Let's first consider the basic form of Figure 3(a) and take the point of view from which the model bulges upwards. Suppose each pleat is $1/16$ wide and we begin folding from the lower left corner and continue until we run out of paper. Every pleat we fold is negative, because the coordinate of its mountain fold is smaller than that of its valley fold. If we begin with a horizontal pleat, and leave two units before it (for the lock), the notation for this first pleat is $-(3/2)h$. The second pleat will be vertical, and written as $-(3/2)v$. The second horizontal pleat will begin two units above the first. The first of these unit lengths is hidden by the previous pleat, and the second is the portion visible in the final model. Thus the second pair of pleats could be written as $-(6/5)h$ and $-(6/5)v$. Writing out the whole sequence is tedious, so we can simplify the notation by introducing an index. The complete folding sequence for the model is the sequence of pairs of the form

$$\frac{-3i}{3i-1}h\frac{-3i}{3i-1}v,$$

where i ranges from 1 to 5 (the largest meaningful coordinate for a 16-unit grid is 16, and 15 is the largest multiple of 3 smaller than this).

Simple bowl. In folding this model, we are effectively interleaving two bulges, one starting from the bottom left and the other from the top right corner. For symmetry, we alternate between folding two horizontal and two vertical pleats. Say we begin with a 32-unit grid. The first horizontal pleat is, as before, $-(3/2)h$. Before folding the corresponding vertical pleat, however, we fold another horizontal pleat at the very top of the sheet, that is, we fold $(29/30)h$. Note that this pleat is positive, because its mountain fold is above its valley. Thus the bowl is completely described by

$$\frac{-3i}{3i-1}h\frac{32-3i}{32-3i-1}h\frac{-3i}{3i-1}v\frac{32-3i}{32-3i-1}v,$$

for i ranging between 1 and 5.

Drawbacks. The simple bowl from the previous example has two mirror lines. (It "almost" has even more symmetries; only the fact that each pair consisting of a horizontal and a vertical pleat cannot be made at the same time prevents this model from having the symmetry of a square. In a sense, it is globally symmetric, but not locally.) However, this symmetry is not really reflected well in the notation. The problem is that we use the same coordinate system for each pleat. In order to give as much information as possible about a model without need for complicated calculations, we might wish our notation to somehow indicate the inherent symmetries (as well as the "almost symmetries", which ignore the ordering of adjacent layers). In the next section we describe such an approach to the notation of pleat tessellations.

6 Compact notation

In most pleat tessellations, each horizontal pleat is immediately followed by a vertical pleat. In other words, we consistently interlace horizontal and vertical pleats. This leads to models that hold together even when folded from paper that doesn't hold creases too well. The tendency of each crease to unfold is kept in check by its immediate neighbors, which at the same time creates the tension that shapes the final fold without need for wet folding. What this means for notation is that perhaps we've gone too far in trying to describe every single crease: if we know the current pleat, the next one is almost completely determined. This observation leads to the next, more restrictive but much more compact and more visual notation.

Pleat sequences made implicit. The basic form consists of a single sequence of pleat pairs, each horizontal pleat preceded and followed by vertical ones. If we know how many pleat pairs we make, we know exactly what the final result is. We ignore the distinction between a left-handed and a right-handed model (determined by whether the first crease is horizontal or vertical). Thus we may as well describe the bulge by just giving the number of pleat pairs to be folded.

Symmetry. To make the symmetry structure of a pleat tessellation apparent, we specify the degree (the number of symmetric pleat sequences folded), 1 (which may be omitted) or 2. The goal is to give meaning to expressions such as 16 (or equivalently 16^1, the basic bulge model) and 16^2 (the simple bowl).

Unit pleats. The width of paper used by a pleat of width 1 is actually 3 units: to cross the width of an unfolded pleat, we first cross a unit in getting to the mountain fold, then turn the corner around the mountain fold and go back a unit width, and finally turn again, and go another unit to emerge from under the pleat. If we are describing the model just in terms of pleats, and they are all of the same width, it makes sense to use the full width of the strip used up by a pleat as a unit of distance. In other words, in the compact notation writing 16 for a grid size will mean that we can fold 15 pleats from the grid.

Example: bulge. This is the simplest case. If there are k horizontal and k vertical pleats, we write just k.

Example: simple bowl. If the sequence starting from the bottom left is seen as a sequence of positive pleats, then so should be the sequence starting from the top right. This model consists of two basic pairs of pleat sequences, and so we write 15^2.

Example: pyramid. Here, the folding begins in the center and progresses towards the edges of the sheet. These are negative pleats, and therefore we write $(-15)^2$.

Example: Two-fold basket. Here, two pleat sequences begin at the centers of two opposite edges of the sheet. Compared to the basic bulge, one of these is rotated 90 degrees. We write $k \cdot i$ for a length k pleat sequence rotated by $\pi/2$ around an axis normal to the plane of the fold. (Note that this is compatible with the use of -1 for negative pleats.) Thus the two-fold basket may be denoted by $(15i, -15i)$.

Example: wave. Here, we are simultaneously folding a bulge from one and an upside-down ("flipped") bulge from the opposite corner. We denote the flip by the letter f, and so the wave is defined by $(15, 15f)$.

Paper size. We may also specify the paper size (in terms of full pleat width units) before the rest of the expression. Thus, a wave folded beginning with a 32×32 grid could be written as $32 : (15, 15f)^2$.

Example: fancy bowl. In this model the folding sequence for the upside-down pyramid is interlaced with the folding sequence for the bowl, resulting in an inverted pyramid in the middle, and four corners whose shape is exactly the simple bowl shape. We denote this bowl by writing $(7, -7f)^2$.

Example: second-order bowl. A second-order pleat consists of two opposite pleats next to each other. We should group the pairs of simple pleats that form second-order pleats: $32 : (7(1, -1f))^2$

More complicated sequences. The basic rule is: recurse and parenthesize. An expression of the form $(A_1)(A_2)$ instructs us to first fold the sequence specified by A_1, then the one by A_2. The pleats of A_2 will be placed in between those of A_1. See Figure 4(d) for the interleaved bowl, defined by $32 : (8)^2(7)^2$.

7 Final comments

Suggestions for folding. The patterns given in this paper may be sufficient to allow experienced folders to reproduce the models. However, in order to make it easier for others, we briefly describe the process. Most of the pieces described in this paper are based on the square grid. The easiest way to fold a grid is to use a power of 2 as the number of squares along each side, and to always divide an existing unfolded strip of paper in two by aligning an edge of the paper sheet with a previously folded crease. Since many parallel folds will stretch the sheet somewhat, it is best to alternate between dividing in half horizontally and vertically. In order to maintain integrity of the paper sheet, it is best to fold only valley folds (or only mountain folds). Our personal preference is to precrease only one half of each pleat, that is, only the mountain fold, and to create the valley fold "on the fly" once the actual folding begins. The main reason for this is that the multiple layers of paper will cause all but the first few pleats to lie slightly away from their theoretical position. If both the valley and the mountain fold are precreased, this "creep" leads to problems. This approach to folding is another factor that makes the compact notation natural, in that it focuses on individual pleats and assumes a regular grid.

Finally, the lock is achieved by simply folding an edge of the sheet twice: two mountain folds in parallel creating the impression of a rolled edge. If the pleats are held tightly as this is done (not easy, and best attempted slowly and only a few pleats at a time), the two folds will prevent the pleats from opening up and will thus lock the edge.

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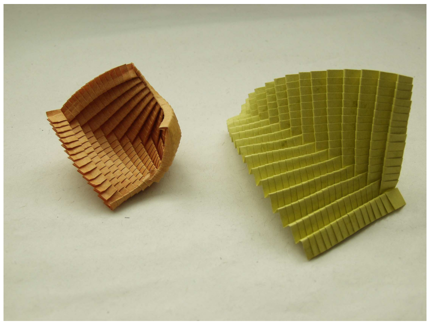
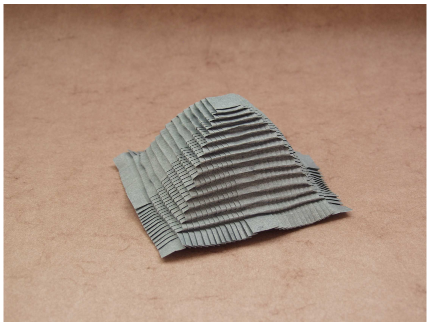
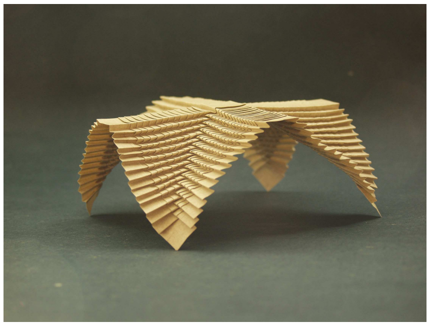


(a) Basic pleat tessellation, locked (the Bulge) and unlocked. 15.

(b) Simple bowl: locked. $(15)^2$.



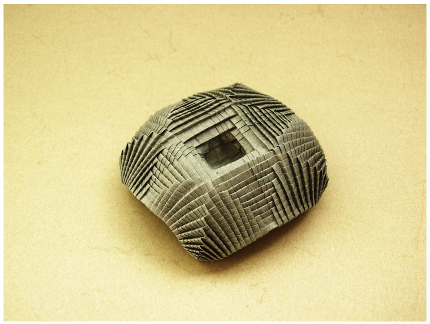
(c) Pyramid. $(-15)^2$.



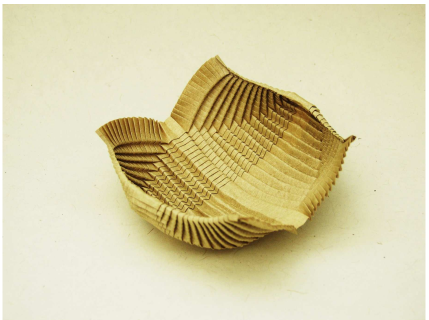
(d) Simple bowl: unlocked. $(15)^2$.

(e) Wave. $(15, 15f)$.

(f) Fancy bowl. $(7, -7f)^2$.



(g) Candy dish. $(10, 5f)^2$.



(h) Two-fold basket. $(15i, -15i)$.

Figure 3: Some pleat tessellations.

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(a) Second order bowl. $32 : (7(1,-1f))^2$.

(b) Valentine. $32 : (15(1,-1f))$.

(c) Double wave. $44 : (11,11f,11,11f)$.

(d) Interleaved bowl. $32 : (8)^2(7)^2$.

Figure 4: More advanced pleat tessellations.