Exploring Some of the Mathematical Properties of Chains

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Abstract

This workshop aims to explore various mathematical topics that emerge from examining classes of chains and their properties. Basic concepts are taken from topology, an area of mathematics that is concerned with notions like connectedness, how many holes there are, and orientability; geometry, including symmetries; and collapsibility and degrees of freedom. These topics are explored through an examination of a small number of chain designs including examples that are not topologically linked at all, examples in which the relative position of the links determine the symmetries, degrees of freedom, and the way in which their structure is analogous to that of a Moebius band, and finally a model of a chain design with a fractal structure. The workshop will include building human models to explore various properties and other activities where the participants will be able to play with necklace models to better understand the theory and to come up with their own questions to investigate.

Introduction

According to a popular online dictionary [1], a chain is 'a series of connected links', and a link is defined as 'one element of a chain' [2]. A more useful way to define a chain is as follows: a series of interconnected, rigid elements called links which, together, constitute a linear, flexible (non-rigid) object that can be used to connect, wind, hang or wrap. Generally, chains are made of a limited number of different link types that combine in repeating patterns, thereby displaying predictable properties such as symmetries.

Some of the mathematical properties of chains, which are explored in the workshop, include the topological nature of specific designs, the symmetries that are present or absent, degrees of freedom, collapsibility and the significance of the ratio of the sizes of the elements, including the eccentricity, gauge (thickness of the wire) and inner radius of the links. These properties are explored through various examples including the not-link chain, the Byzantine chain design, the 'Rope' chain design and Antoine's Necklace [3].

The Not-Link Chain

In knot theory, a 'link' refers to a 'set of knotted loops' or circles [4]. We call the first example under consideration the 'not-link chain' because although it functions as a chain according to the general definition given above, in knot theory terms it is not linked as the elements are not looped through each other.

Figure 1: *The Not-Link Chain*

In Figure 1 on the left, a section of the chain shows how it alternatively connects two different types of elements: the first is a rod at either end of which is attached a slightly larger metal ‘ball’, forming an elongated dumbbell; the second element is a small flat figure eight whose holes are of a smaller diameter than the ‘ball’ of the dumbbell but larger than that of the rod. These size ratios allow some movement in the chain, and prevent it from disconnecting. The right side of Figure 1 shows a connection between the two types in more detail.

The Byzantine Chain

The second example under consideration is known as a Byzantine Chain, King’s Chain or Bali Chain [5]. It is related to a design called Idiot’s Delight [6], and is made using only one type of element: a simple round ring. Figure 2, below, shows a section of Byzantine Chain made using shower curtain rings. The distinguishing feature of this design is the interlocked 4-ring combination in the circle. This grouping is made rigid by the addition of the rings connecting it to the next interlocked 4-ring grouping.

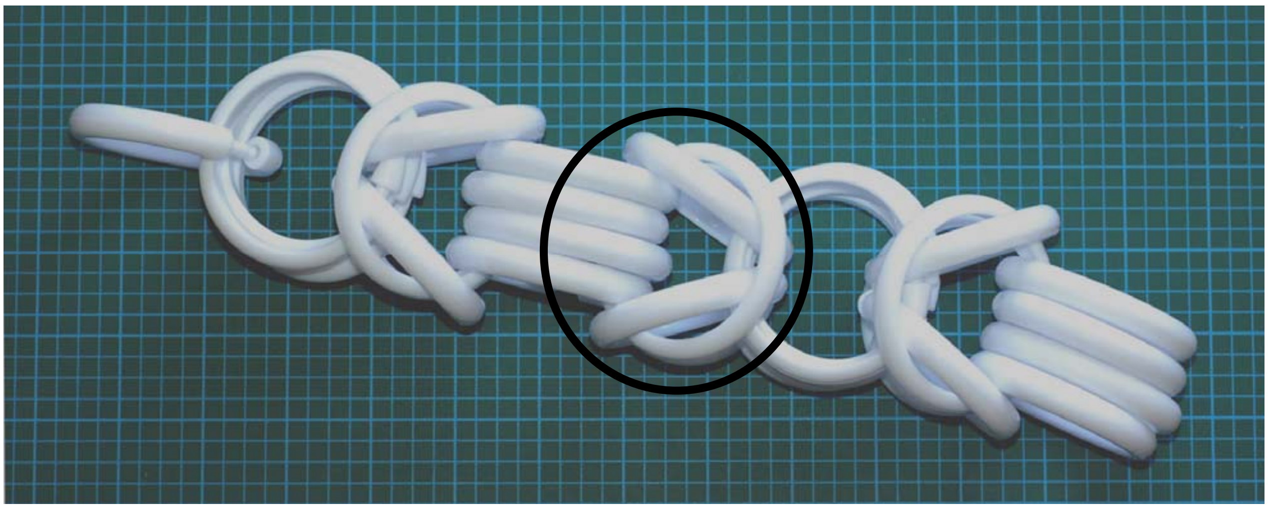

Figure 2: *Three Sections of Byzantine Chain*

The design of the Byzantine Chain is interesting from a topological point of view in terms of the interlocking of the elements. It is also a good example for examining three-dimensional symmetries. In addition to longitudinal translation symmetry, the chain also has reflection symmetry across two perpendicular, longitudinal planes (one parallel to the picture plan, one perpendicular to it) and one type of transversal plane (through each set of four aligned elements). It also has two kinds of ‘glide rotation’ symmetries. Firstly, locally, the interlocked grouping can be rotated 90^o around the chain’s longitudinal axis, then reflected through the transversal plane through its center, and the global positions of the chain are preserved. Secondly, the grouping can be rotated through 90^o again, then translated to the position of the next grouping and globally, the positions are all preserved.

This design is often executed in silver, in a much more compact version than in the figure above. In this case, the links that form the interlocked groupings are oval, only two links are used between the interlocking groupings instead of four, and the chain acquires a more pronounced rigidity [5]. The degrees of freedom are then reduced to rotations around axes located in the centre of the connecting pairs of links. Each alternate joint can only move in one rotational direction, perpendicularly to its neighbours, and the chain is much less fluid.

The 'Rope' Chain

The third example under consideration is a simpler design in that every element is in the same position relative to the others. In tiling terms it is 'monohedral' [7]. In the Rope Chain design, a standard chain design is modified in such a way that every element is connected, not only to its immediate neighbours on either side, but to two or even three. In Figure 3, the elements connect four others each time.

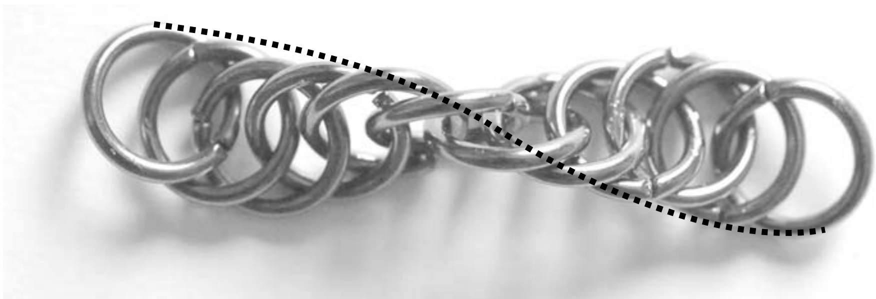

Figure 3: *The Twisted Rope Chain Design*

This design also has several symmetries, including rotation, longitudinal translation and 'glide rotation' through which each element is related to all the others. It has no global reflection symmetry. In the case of the design as executed in Figure 3, the elements are still loose and the chain can be untwisted somewhat so that the staggered positions are not as visible. This flexibility depends on the relationship between the gauge and the inner radius of the rings. As the gauge is increased relative to the inner radius, the chain becomes more rigid and the 'twist' more stable. At the extreme, the flexibility comes solely from the freedom of movement afforded by the rotational symmetry of the individual elements that can slide around each other.

The most interesting feature of this design, from a topological perspective, is the twist built up in the chain. If a line is traced that connects the outermost point of each element, on either side, as shown in Figure 3, then the design can be seen as a band or ribbon that twists around itself and whose ends can be joined, *just like a Moebius band*. The gauge, the inner radius and the number of links that are interconnected together within the length of the chain will determine how many times the 'band' twists around itself and therefore whether the resulting object has one or two edges and one or two faces.

Antoine's Necklace

Antoine's necklace is an interesting topological space that is defined recursively [3]. Start with a torus V. C_1 is a chain of tori linked together as in Figure 1. In each component of C_1, construct a smaller chain of tori (generally with the same number of links as in C_1). Let C_2 denote the union of the smaller tori at this level. Continue this process *ad infinitum*, and Antoine's necklace is the intersection of the C_i. This set is a non-empty, compact subset of $\mathbf{R}^3$. In $\mathbf{R}^3$, being compact is equivalent to being closed (contains its boundary) and bounded. Each C_i is a subset of C_{i-1} (for i greater than or equal to 2), so the intersection of the nested sets is non-empty.

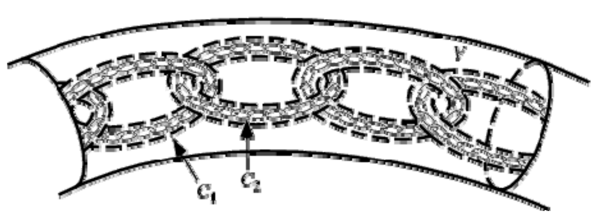

Figure 1: *Antoine's Necklace (http://mathworld.wolfram.com/AntoinesNecklace.html)*

Antoine's necklace is a fractal that is homeomorphic (or topologicially equivalent) to the middle thirds Cantor set [3]. The middle thirds Cantor set can be obtained as follows [8]. Start with a closed interval of unit length, remove the middle third open line segment leaving two closed line segments each of length one third. Then remove the middle thirds of the remaining closed line segments. Continue this process *ad infinitum*. The resulting set possesses many interesting properties. It is compact, uncountable (infinite but cannot be put in one-to-one correspondence with the natural numbers), and disconnected (given any two distinct points x and y in the set, there exist two disjoint open sets U and V such that x is in U and y is in V). The set is also self-similar, and one can find the corresponding similarity dimension D. An example of a more familiar self-similar object is the square. The similarity dimension D of the square is 2 because scaling down by a factor of 2 gives 4 similar squares, thus $2^D=4$. In the Cantor set, the original line segment is scaled down by a factor of 3 to obtain 2 similar line segments, so D is given by $3^D=2$, or $D = \log 2/\log 3 \approx 0.6309$.

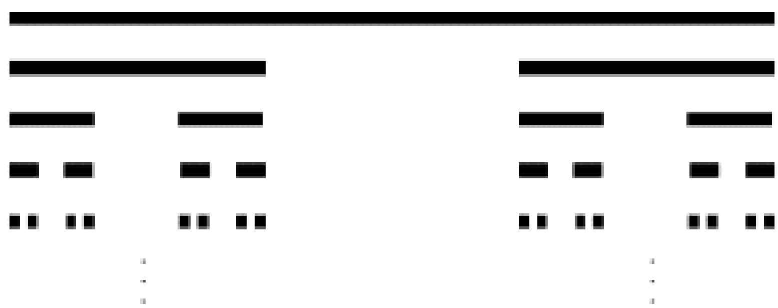

Figure 2: *Middle Thirds Cantor Set*

Conclusion

In summary, the workshop will introduce participants to different mathematical concepts that arise from studying various classes of chains. The hands-on activities will allow participants the opportunity to see and feel the abstract concepts and to come up with their own mathematical discoveries and queries about the chains.

References

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