

## Mathematical Experiments with African Sona Designs

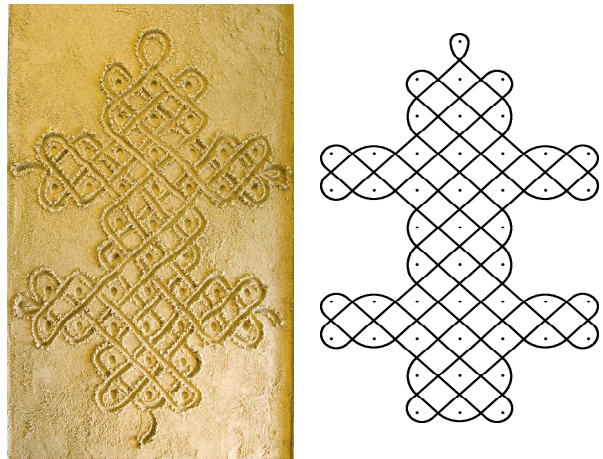
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### Abstract

The sona designs of the Chokwe people of Angola/Congo are a particularly attractive form of “mirror curves,” which can be visualized as a curve drawn through a lattice of dots, bouncing off the edges of the lattice and off “mirrors” placed between some of those dots. Most such designs are drawn as a single, uninterrupted line, and most contain symmetries of some type. The mathematics of the designs reflect issues of common divisors of numbers, Eulerian cycles, and symmetry groups. This talk will investigate ways to use a sona-drawing program to investigate these topics with students from middle school through college, with particular emphasis on the experimental discovery of mathematical facts. The sona program developed by the author is cross-platform and free.

### Introduction

The Chokwe people of Angola and Congo have a drawing tradition, done both in sand drawings and on more permanent objects, that has attracted a substantial amount of mathematical attention (e.g. Ascher [1], Gerdes [4], Jablan [5]). Although their sona (singular “lusona”) drawings arise in several different forms, one of the more common, and the most mathematical, can be viewed as a grid of dots with a curve passing through the grid, “bouncing” off the boundary and off internal “mirrors,” to create a single continuous curve. The “Leopard with Cubs” lusona shown in figure 1 is an example of such a drawing without internal mirrors. This sand drawing includes a few features added to the fundamental design to indicate the heads and tails of the mother leopard (vertical) and her two cubs (horizontal). Figure 1 shows it in three forms: a photograph of the drawing made in the sand, a computer rendition of the drawing showing it with the heads and tails of the leopards, and a computer drawn form of the underlying Eulerian drawing, as the sona drawing program draws it, without the artistic additions to personify the animal forms the artist has seen in the principal monolineal drawing.

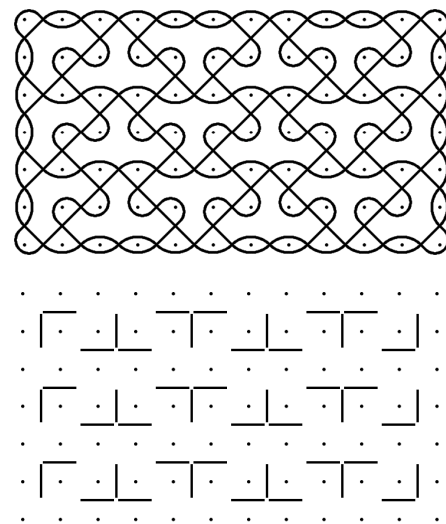


**Figure 1:** *The “Leopard with Cubs” lusona drawn in the sand, with additional heads and tails of the mother leopard and her two cubs. On the right is the underlying monolineal curve, as drawn by the sona program.*

The “Leopard with Cubs” is a sona example with no internal “mirror” walls for the curve to bounce off. A classic example of a sona that uses such mirrors is the “Chased Chicken” of figure 2, so called because it seems to capture the path that a chicken might take if you were trying to catch it.

Figure 2 shows both the sona as it would be drawn by a Chokwe artist, and the “hidden” walls that help to define the shape. The artist making this drawing lays down the grid of dots shown, does *not* draw the walls, but draws the Chased Chicken sona directly on the dot grid. There is some evidence that the Chokwe artist envisions the walls as they draw the sona. In many cases, the wall layouts help us to see more clearly the symmetries within the sona design. Both parts of figure 2 were drawn by the sona program.

The range of sona designs gives rise to many types of investigations, both mathematical and artistic. The use of the computer program to try different layouts of dots and walls allows a student to imitate what might be months worth of work by a Chokwe artist looking for interesting designs. A fundamental challenge is that not all grids lead to acceptable sona drawings: they may not be monolineal, or to make them monolineal may require extra walls that disrupt the symmetry or aesthetic appearance of the design. Our experience, with students from 4th grade through college, is that these experiments and open-ended design questions are enjoyed greatly by the students, many of whom continue to work on designs well after the assignments have been completed.



**Figure 2:** The “Chased Chicken” sona, on a 7 X 12 grid, along with the mirror walls that define the curve. The design has  $pmg$  symmetry.

### Some Mathematical Investigations

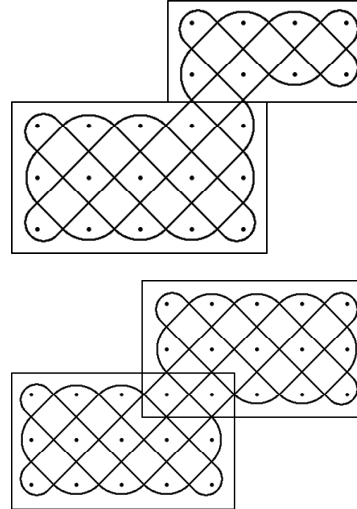
**Greatest Common Divisor Investigations.** A basic experiment with sona is to determine for what dimensions an  $m \times n$  rectangle, with no walls, will give a monolineal sona. The answer is “when  $m$  and  $n$  have no common factor,” i.e. when  $\gcd(m, n) = 1$ . In most cases, the students’ original hypothesis is that it happens whenever one of  $m$  and  $n$  are odd. A little experimentation, occasionally with suggestions of rectangles to try, generally leads students to the correct answer. (For young students, this is especially true if they have recently been simplifying fractions.) A similar investigation is to ask “If a rectangle does not give a monolineal sona, how many lines does it take to draw the sona?” (Answer: It takes  $\gcd(m, n)$  lines.)

Some African sona designs are built from combinations of rectangles, such as *abutting rectangles* or *overlapping rectangles*, as shown in figure 3. This leads to easy extensions of the basic rectangle investigation just mentioned. For example, we can ask “When do rectangles abutting (i) along a single dot, or (ii) abutting along two dots, give monolineal sona?” (Answer: (i) If both rectangles have dimensions with gcd’s of 1; (ii) If one rectangle has a gcd of 1 and the other has a gcd of 2.) Similar, but slightly more complicated answers, apply to rectangles overlapping by 1 or 2 dots.

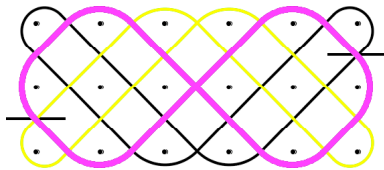
**Additional GCD Investigations:** The exercises below have answers that can be discovered through experimentation, and which are simply enough to be found empirically. As with the examples above, these investigations are usually too tedious to do by hand, but fairly easy to do with computer assistance from the sona drawing program. Additional exercise are online at <http://math.beloit.edu/chavey/Sona>.

- 1) If a single dot is erased from a corner of a rectangle, when will the resulting sona be monolineal? (Answer: If, and only if, the rectangle dimensions have a gcd of 2.)

- 2) If a rectangular grid has dimensions with a gcd of  $r$ ,  $r > 1$ , can you erase dots along one edge, starting from the corner, to create a layout with a monolineal sona? If so, what is the smallest number of dots you need to erase? (Answer: Yes, by erasing  $r-1$  dots.)
- 3) If two rectangular grids with gcd's of  $r$  and  $s$  respectively are aligned so as to overlap at one corner dot, can you tell how many lines the resulting sona will need? (Answer:  $r + s - 1$ .)
- 4) If you have two identical rectangular grids, with a gcd of  $r$ , can you overlap them in a  $1 \times n$  rectangle so the resulting sona is monolineal? (Yes; they should overlap in a  $1 \times r$  rectangle.)
- 5) By starting with a monolineal rectangle sona, and then attaching squares of dots, what interesting sona (e.g. animal or naturalistic shapes) can you construct? Look at examples of authentic Chokwe sona for ideas of shapes that might be appreciated by those artists.



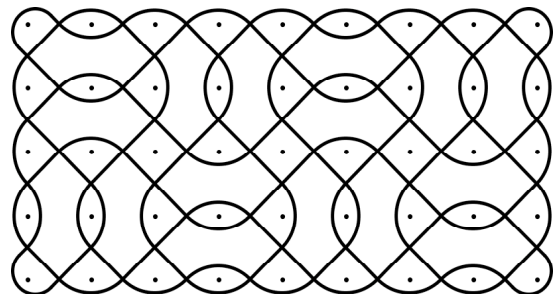
**Figure 3:** Monolineal sona built from abutting and overlapping rectangles. On the top, two rectangles “abut at two dots,” while on the bottom they “overlap at two dots.”



**Figure 4:** The tri-lineal sona constructed on a  $3 \times 6$  rectangle grid. This sona can be made monolineal by many different choices of 2 symmetrically placed mirror walls.

**Mirror Investigations.** An important result about mirrors (see [3]) is that if a sona with one or more lines has a mirror added where two different lines cross, those lines will merge into one, while adding a mirror where a line crosses itself splits that line into two. The sona drawing program draws multiple lines in separate colors, so it’s easy to locate places where adding mirrors will reduce the number of lines. We can use this idea to construct monolineal sona while meeting the Chokwe symmetry aesthetic. For example, in figure 4 there are many ways to find symmetric placements of two walls so that each wall merges two of the three colors, resulting in a monolineal sona. One such placement, with rotational symmetry, is indicated. In open-ended investigations, students can construct a tentative sona, then use this technique to change it to be monolineal.

**Symmetry Group Investigations.** To imitate the Chokwe aesthetic, we want to build symmetric sona. Many authentic sona have bilateral or rotational symmetry, although often with disruptions to the symmetry from adding animal or natural features. Creating such symmetric sona is an enjoyable investigation by itself. One particular area for interesting, open-ended artistic exploration is the construction of sona with wallpaper pattern symmetries. The Chased Chicken design of fig. 2 is one such example. Figure 5 shows another. Several other examples can be found in Chavey [2]. This suggests several options for investigation of new designs that the Chokwe would likely appreciate:



**Figure 5:** A monolineal sona with  $p4g$  symmetry. This sona uses a wall motif of the form:  $| \cdot |$  This design will be monolineal for all dimensions of the form  $(4m+1) \times (4n+1)$ .

- 1) The design of figure 5 uses a pair of walls ( | | ) as a building block to construct a 2-dimensionally symmetric pattern, while the Chased Chicken design uses a T shape (truncated on the sides) to construct such designs. What types of simple building blocks can we use to construct such designs? Some students have found examples answering this in 40–60 minutes, but I think of this type of question as part of a longer exploration. I generally have students work in pairs.
- 2) For a given building block (a.k.a. “fundamental region”), what wallpaper symmetry groups can be applied to that design to give monolineal sona? With what types of spacing between the wall motifs? And for what sizes of rectangles will that building block and symmetry group generate such designs?

### The Sona Program

The sona program itself allows the user to place rectangles of dots in a grid on the screen, and construct more complicated layouts from unions of such rectangles. A *wall tool* allows the user to place walls at any location within the grid layout. The *draw tool* then initiates the process of drawing a sona line from a user-specified point as defined by the grid layout and the walls. If a sona configuration requires multiple lines to complete the sona, each successive line will be drawn in a new color (within limits). The scale of the drawing, the pen width, the speed of the drawing, and whether the walls are shown or hidden are each adjustable by the user. Designs can be printed or saved to a file, e.g. to be turned in to the instructor. The instructor can also use these files to demonstrate interesting sona, without constructing them in real time, or as starting points for student investigations. The program is written in Java, hence should be fully cross-platform, and is freely available from <<http://math.beloit.edu/chavey/Sona>>.

### Summary

Sona are useful designs to show students at many levels some of the interactions between mathematics and patterned artwork. They allow students to do both directed experimental investigations, or more open-ended investigations. Explorations topics include common divisors, the Euclidean algorithm for common divisors, and implementations of many types of symmetry groups. The students are challenged to find patterns that would be appreciated by the Chokwe artists, while using the mathematical properties to direct their explorations. Many years of experience by the author, and others, with the software used to construct these sona show the topic to be interesting and exciting to students of various ages, including some who have discovered original results through its use. We will demonstrate several of these explorations in the process of showing the functions and capabilities of the Sona drawing software.

### References

- [1] M. Ascher, *Ethnomathematics*, Wadsworth, Inc., Belmont, California, 203 pp, 1991.
- [2] D. Chavey, *Symmetry Groups of Chokwe Sand Drawings*, to be submitted to *J. of Math & Art*.
- [3] D. Chavey & P. Straffin, *The Analysis of Chokwe Sand Drawing Grids*, to appear.
- [4] P. Gerdes, *Sona Geometry Vol. 1*, Maputo, Mozambique: Ethnomathematics Research Project Instituto Superior Pedagógico, 200 pp, 1994.
- [5] P. Gerdes, *Sona Geometry from Africa*, Polimetria, Milano, Italy, 2006.
- [6] S. Jablan, *Mirror Curves*, <<http://members.tripod.com/~modularity/mirr.htm>>.