A Group Portrait on a Surface of Genus Five

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Abstract

This paper represents a finite group with 32 elements as a group of transformations of a compact surface of genus 5. In particular, we start with a designated pair of regions of this surface, and each region is labeled with the group element, which transforms the designated region into it. This gives a portrait of that finite group. These surfaces and the regions corresponding to the group elements are shown in this paper. William Burnside first gave a simple example of such a portrait in his 1911 book, “Theory of Groups of Finite Order”. This paper is the third paper in a series which models groups as groups of transformations on a compact surface in the style of William Burnside.

Introduction and Historical Perspective

A group is a set and an associative binary operation which contains an identity such that each element has an inverse element in the group. Therefore, a group is an abstract object. Groups were originally thought of as permutations of some other mathematical structure, such as a set of points. This permutation group idea comes very naturally from the set of symmetries of physical objects. Thus the symmetries of an equilateral triangle are a group with six elements. It follows that groups can be both abstract objects and real physical motions of a symmetric object.

A group can also be thought of as a set of transformations of a "plane" into itself that is closed under composition. Some groups of transformations can be realized on the Euclidean plane and these give rise to the tessellations of the plane. Many groups need to be realized on the hyperbolic plane and give rise to tessellations of it. A finite group is a finite set of transformations which is closed under composition. If a finite group were represented on an infinite plane, then the fundamental region would have to be infinite. Therefore, a finite group is represented as a set of transformations of a compact two dimensional surface, such as the sphere or the torus. These surfaces must become more complex in order to contain the portraits of some groups.

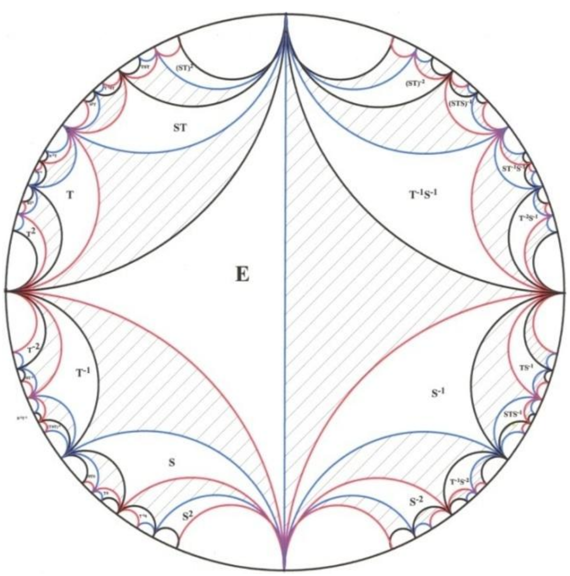

Figure 1, Portrait of a Free Group

This paper is part of a series of papers on how to draw a portrait of a finite group, in the style of Burnside [1]. Burnside started with circles in the plane and used inversion in the circle as the transformation. The relationship between circles determines the group generated by these transformations. When we look at the regions this generates inside of a circle we get Figure 1. Burnside [1, p. 379] constructed a free group on 2 generators, F_2, using two mutually tangent circles and the line tangent to them (a circle centered at infinity). We only need to picture the part of each region inside the circle through the three points of tangency. Like Burnside, the initial region is the "triangle" labeled E and its corresponding shaded region. Each “triangle” is bounded by arcs colored red, blue or black in our sketch. Inversion in any single arc will take a shaded region into a non-shaded region and vice- versa. Each region can be labeled by the transformation needed to get from E to that region. Since we are interested in orientation-preserving transformations, each group action is represented by the composite of two such inversions. Inversion through first a red arc and then a blue arc corresponds to multiplying on the left by the generator S. Multiplying on the left by the generator T corresponds to inversion through black and then red. Multiplying on the left by ST corresponds to inversion through black and then blue. If we considered inversion through a black arc first and then a blue arc as the inverse of a single generator, R, then we could interpret this picture as a portrait of a group with presentation $T_2 = \langle R, S, T \mid RST = 1 \rangle$. This construction fills up a unit disk with black and white regions and the transformations are given in the same way. We have used Geometer’s SketchPad [4] to reconstruct this portrait of a free group on two

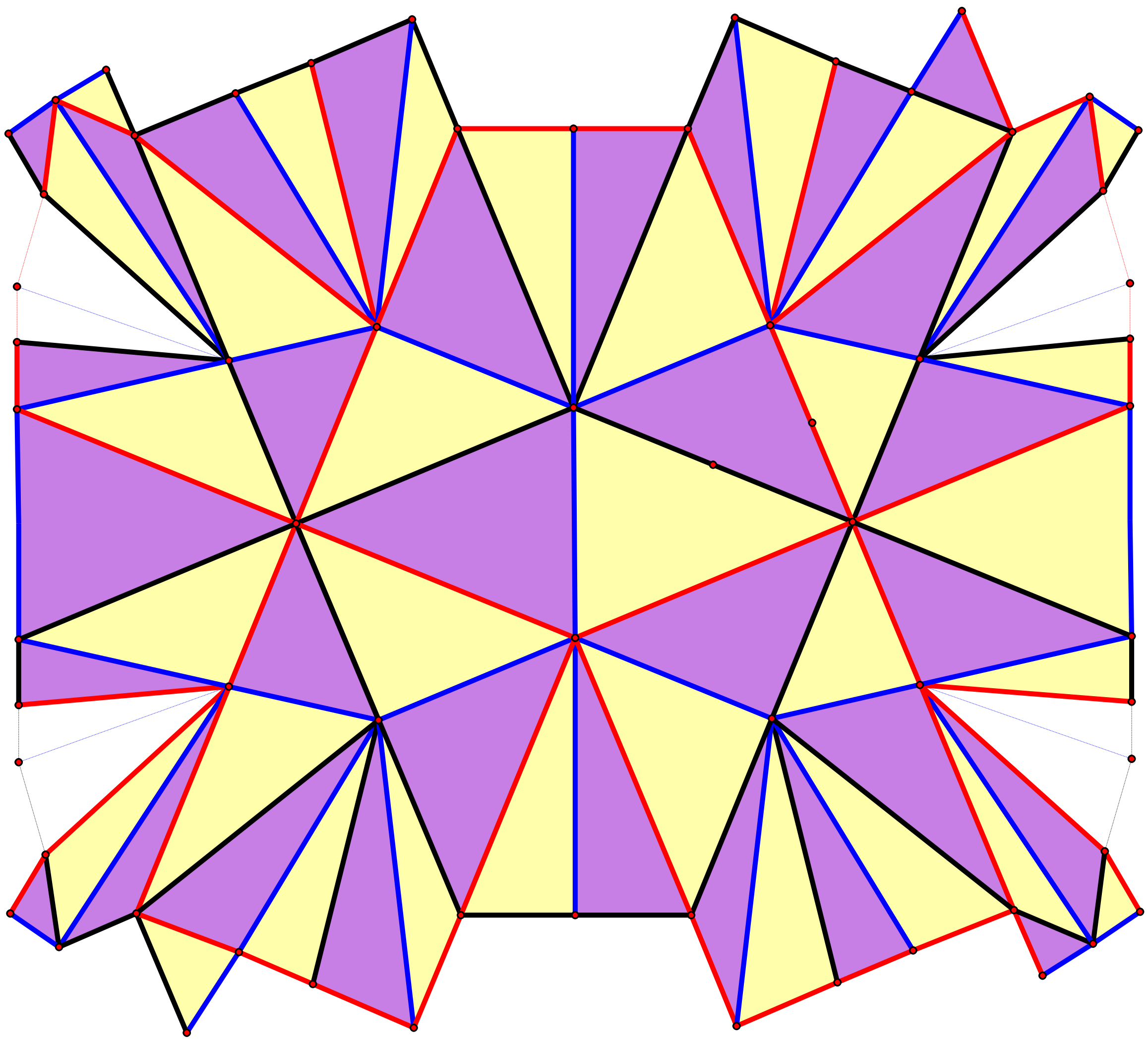

Figure 2 - The Fundamental Region for SG(32,2)

generators (Figure 1), similar to a figure in Burnside [1, p. 380].

Now suppose that we have a finite group, G, generated by 2 generators. The group G is the image of F_2 by a normal subgroup N. Specifically, two strings of generators represent the same element of the group if the product of one string and the inverse of the other string is in the subgroup N. One example of this is that a rotation of 120º clockwise is the same as two rotations of 120º counterclockwise. After associating an element of F_2 to each region, the final step is to identify all regions with labels from the subgroup N. After this identification, we have the finite group G. However the circle in Figure 1 still has an infinite number of regions labeled with elements from the presentation T_2. This circle can be thought of as the Poincare disk model of hyperbolic space. At this point, we choose a connected set of regions which contains a single region for each group element and whose label corresponds to that group element. This set of regions is called the Fundamental Region for the finite group. Any region outside of the fundamental region is equivalent to some region within the fundamental region. Therefore, the compact two dimensional surface is derived by folding the fundamental region in certain ways. This is the same idea as constructing a torus by taking a rectangle and identifying the top and bottom as connecting to each other and the two sides as connecting.

In the diagram in Figure 1, each element has infinite order and the curvilinear triangles get smaller and smaller as they approach the boundary of the circle. This means that the connected region in Figure 1 would have a ragged boundary with parts of the boundary intersecting each point. This can be fixed by picturing a different tiling of the hyperbolic plane or by drawing a polygonal region with the same relationship between the triangles. The second approach has the advantage that the triangles remain easily visible as they get near the boundary of the polygonal region and this portrait is pictured in Figure 2. The portraits developed are topologically equivalent to the model that we want, but even the areas of the regions are changed.

Compact surfaces are classified topologically by genus and orientability. Every compact orientable surface with genus g is topologically equivalent to a sphere with g handles. Very roughly, the genus is the number of "donut" holes that a surface has. This is why a donut and a coffee cup are topologically equivalent. Thus, every compact surface of genus g may be constructed in many different, but topologically equivalent ways. We will pick a symmetric way of representing the surface and use it. This surface may be drawn and colored with each face composed of one white and one black region. These faces represent a finite group of transformations, which act on the surface in the style of Burnside [1]. The choice of surface is made arbitrarily with the correct orientability and genus.

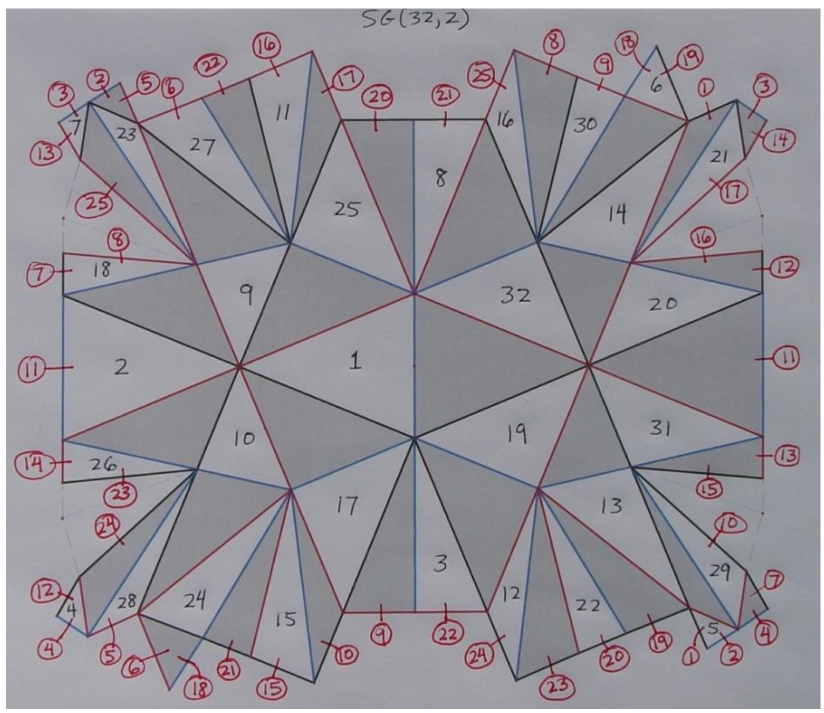

Figure 3 - Connectivity of the Fundamental Region

There have been several Bridges papers which pictured regular maps on surfaces of genus 3 to 7 (for example [3] and [7]) and their associated tilings and the related topic of portraits of groups using techniques in this paper ([9] and [10]). The automorphism groups of these tilings are the groups PSL(2,7) (in [3]) and S_5 (in [7]). The portraits in [9] are of the dicyclic group of order 12 and the group of order 16 with notation < 2, 2 | 2> (see [2, p. 134]) acting as orientation-preserving transformations and the group P_{48} = SG(48, 33) pictured as a group where some transformations are orientation-reversing.

Group Portraits

Let G be a finite group. There is a set of orientable surfaces on which G can be represented as a group of transformations. One of these surfaces has the smallest possible topological genus of all surfaces in this set. The genus of this surface is defined to be the symmetric genus of the group. If the group can only be represented by transformations that preserve the orientation of the surface, then the genus of the surface of smallest genus is called the strong symmetric genus of the group.

The group that we will be considering is a group of order 32 with symmetric genus 5 (see [6]). The genus action is given as a quotient of the triangle group $\Gamma(4,4,4) = \left\langle S,T \mid S^4 = T^4 = (ST)^4 = 1 \right\rangle$ and these transformations preserve the orientation of the surface, its strong symmetric genus is also 5. This group is SmallGroup(32,2) in the Magma Library of groups (see [5]). We denoted this group by $SG(32,2)$. It has a presentation $\left\langle S,T \mid S^4 = T^4 = (ST)^4 = ([S,T])^2 = [S,[S,T]] = [T,[S,T]] = 1 \right\rangle$ as the image of $\Gamma(4,4,4)$. This presentation is extremely symmetrical. We know that this group can be drawn as a group of transformations of a compact surface with five "donut holes" and that it cannot be

Figure 4 - The model of SG(32,2) on a surface of genus 5

drawn on a compact surface with any fewer than five "donut holes". The polygonal region for the group $SG(32,2)$ is given in Figure 2.

In Figure 3, each of the thirty two faces is labeled with a number which corresponds to a group element. Remember that each "face" is a paired white and black region separated by a blue curve. Each region on the border of the plane figure connects to another region on the border of the plane figure. Theoretically, this tells us how to fold the fundamental region to get a compact surface with genus five. The faces are split into white and black parts simply because this is the way that Burnside constructed his original diagram. This portrait consists of 32 white and 32 black triangles. Each of the 64 regions is bounded by 3 edges and each edge bounds 2 regions. So a simple combinatorial argument gives that there are 96 edges. The faces that meet at a vertex are labeled in such a way that each face is related to the adjoining face by multiplication on the left by either S, T, ST or its inverse. For example, two regions M and N are both incident to the same S vertex if and only if $N = S^k M$ for some k. Therefore, each vertex could be classified as an S-vertex, a T-vertex or an ST-vertex depending on the labeling of its bounding regions. Since S, T and ST have order 4, each S, T or ST vertex has degree 8. This gives 24 vertices. Therefore, the Euler characteristic is χ = V - E + F = –8. Since χ = 2 - 2g, where g is the genus of the surface, this portrait is drawn on a surface of genus five. Therefore, the portrait in Figure 4 gives the genus action for this group (see [6]).

Finally, we construct a model of the surface and of the transformations in this group. This model is in Figure 4. Notice that this is a model where each transformation is represented as a word in the generators of a particular presentation. Therefore, one could say that this is a model of the particular presentation of this group. I would point out that this presentation was chosen as the one which can be drawn on a surface of smallest genus. In this particular case, the Cayley graph of this presentation is the dual of this model where each region consists of a paired white and black region. However, in general Cayley graphs can be embedded in surfaces of smaller genus than can be done with this technique (see [8]).

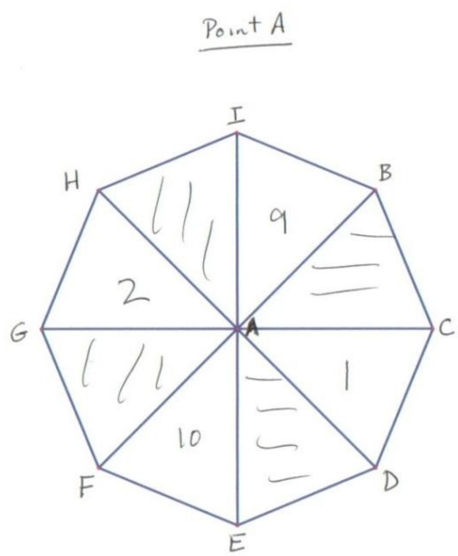

Figure 5 - Local information about a point

Construction of the Model

Finding a blueprint for the model is the hardest part of this process. A graph is constructed from the data in figure 3. This graph is a regular map, but unfortunately this fact is not useful in finding the model. The basic idea is to construct a model of a genus five surface and try to fit the points of the graph on the model in the proper way. The first step is to get the information about the points and faces that are adjacent to a point. An example of this is shown in Figure 5 for the point labeled A. Diagrams like this one are constructed for each point in the graph. Notice that although Figures 2 and 3 distort the angles, in the final model the triangular regions which have point A as a vertex, all have vertex angle 45º. In Figure 5, the points H, A and D are on a straight line and in the model this is a black curve. We can extend this "straight line" to a closed curve on the model. These curves are referred to as "straight line curves".

There are 24 straight line curves in the model, eight each of black, red and blue curves. Each straight line curve contains four vertices. Each vertex is on four straight line curves, two of one color and two of another color. The two straight line curves of the same color which intersect in a point are at right angles. If you start at any point in the model and pick a color incident to that point, the two straight line curves intersect in exactly two steps. For example, the two black straight line curves incident to point A are H $\rightarrow$ A $\rightarrow$ D $\rightarrow$ L and F $\rightarrow$ A $\rightarrow$ B $\rightarrow$ L and they intersect again at point L. The two red straight line curves incident to point A are G $\rightarrow$ A $\rightarrow$ C $\rightarrow$ P and E $\rightarrow$ A $\rightarrow$ I $\rightarrow$ P and they intersect again at point P.

Two white regions M and N both bound a straight line curve of the same color if and only if

$N = S^2 \cdot M$. The same is true for T and ST vertices. Thus, a straight line curve contains directed edges that are two steps apart if and only if these edges are incident to the white regions F and x^2y^2F where $x, y \in \{S, T, ST\}$ are different generators. Two steps preserve the direction of the directed edge along the straight line curve. The elements S^2, T^2 and $(ST)^2$ are in the center of $SG(32,2)$. So if $N = z^k \cdot M$ for $z \in \{S, T, ST\}$, then both images x^2y^2M and x^2y^2N have the same initial vertex because they satisfy the equation $x^2y^2N = x^2y^2z^k \cdot M = z^k x^2y^2M$. This is why the straight line curves above always intersect in 2 steps. It is also clear, since these elements have order 2, that four steps gets you back to the initial point.

There are 24 points and since there are 5 holes and the outer rim, each one of these has a straight line curve on its rim. Now we find a way to connect these points. There are in fact multiple ways to do this, but one constraint used is that the front and back side of the models must be symmetrical. This gives a pleasing symmetry to the whole construction and makes it easier to see how it fits together.

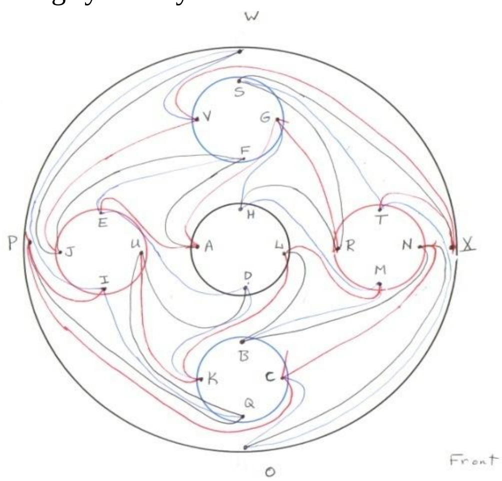

Figure 6 - Blueprint of the Model


Figure 7 - Model with graph on it.

Finally, the front face of the blueprint for the model is included in Figure 6, and the actual model with the graph drawn on it is in Figure 7. Note that even though the graph was embedded on the model first, it is easily checked that the faces are properly positioned.

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