A New Method for Designing Iterated Knots

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Abstract

A new method for designing iterated knots is described. The method utilizes a starting curve in which smaller copies of the curve are embedded. Points along the curves are connected by strands to form a knot, which is iterated by replacing the contents of the smaller curves with the contents of the larger curve. The method possesses considerable design flexibility, allowing the creation of stylized and esthetically pleasing iterated knots.

1. Introduction

Fractals and knots are popular topics at Bridges Conferences, both being mathematical objects with considerable visual appeal. In 2005, Carlo Séquin showed a couple of examples of iterated knots, created by applying the iterative substitution concept of fractals to knots [1]. The basis for this approach is Conway's notion of a tangle, a portion of a knot isolated by a boundary that is crossed four times by the strands [2]. Descriptions of knots and operations on knots are facilitated by the use of tangles.

At Bridges 2007, Robert Fathauer presented a paper that described a widely-applicable method for creating fractal knots by iterative substitution [3]. In that method, the starting point is a knot from the Knot Table [2]. The knot is rearranged to have holes that are similar in shape to the overall knot. It is then iterated by substituting copies of the current knot for the holes. While it is relatively straightforward to apply this method to any knot in the Knot Table, the method does not allow much flexibility in guiding the design to result in esthetically pleasing knots.

In this paper, a new method is described that allows greater flexibility in the esthetics of the design. The starting point is not the Knot Table, however. If desired, one can work backwards from the chosen starting knot to determine the knot in the Knot Table to which it corresponds. The work presented here is a recreational and artistic endeavor that makes use of established knot theory.

2. Description of the Method

The method is described below and illustrated in Figure 1. More concise and rigorous mathematical language could be used, but the language chosen should be accessible to a wider audience.

1. Create a "starting knot interior" as follows:
   a. Create a simple closed curve C_0 within the plane.
   b. In the interior of C_0, embed one or more smaller curves C_1^1, C_1^2, ... that are similar to C_0 within a set of operations that include scaling, rotation and reflection, as desired.

c. Within each C_1^i, draw strands with one or more crossings, terminating at points P_1^{i1}, P_1^{i2}, … on C_1^i. Each group of strands and points should be similar (identical except for the differences in scaling, rotation, and reflection used in Step 1b). There should be at least four points P on each curve, and the number of points must be even. I.e., each curve bounds an n-strand tangle with $2n$ endpoints.

d. Create a set of points on C_0 that is similar to the set of points on each C_1^i. Connect the loose strands terminating at the points P_1^{i1}, P_1^{i2}, … on the each C_1^i either with points on other C_1^i or with the points P_0^1, P_0^2, … on C_0.

2. Outside of C_0, join the loose strands terminating at points P_0^1, P_0^2, … in such a way as to obtain a unicursal (single-strand) knot with alternating over/under weave. These strands, the "exterior strands", together with the starting knot interior, constitute the starting knot. Check to be sure that this starting structure really is a knot; i.e., that it is unicursal and that it isn't actually the unknot (that it can't be reduced to a simple loop with no crossings). The signs of some of the crossings in one or more of the C_1^i may need to be flipped in order to maintain an overall over/under weave.

3. In the starting knot interior, replace each C_1^i and it's interior with a scaled-down copy of the starting knot interior, created according to the sets of operations used in Step 1b. Check to be sure that this new knot is unicursal. Change the sign of the crossings with each C_1^i as needed to maintain an overall under/over weave when the exterior strands are included. Call the resulting structure, exclusive of the exterior strands, the "current knot interior".

4. Replace each C_1^i and it's interior with a scaled-down copy of the current knot interior, created according to the sets of operations used in Step 1b. Check to be sure that this new knot is unicursal. Change the signs of the crossings with each C_1^i as needed to maintain an overall under/over weave when the exterior strands are included. Call the resulting structure, exclusive of the exterior strands, the "current knot interior".

5. Repeat step 4 as desired. After all repetitions, the strands defined in Step 2, together with the current knot interior, constitute the final iterated knot. The exterior strands may be modified if desired, as may the strands within the smallest generation of curves.

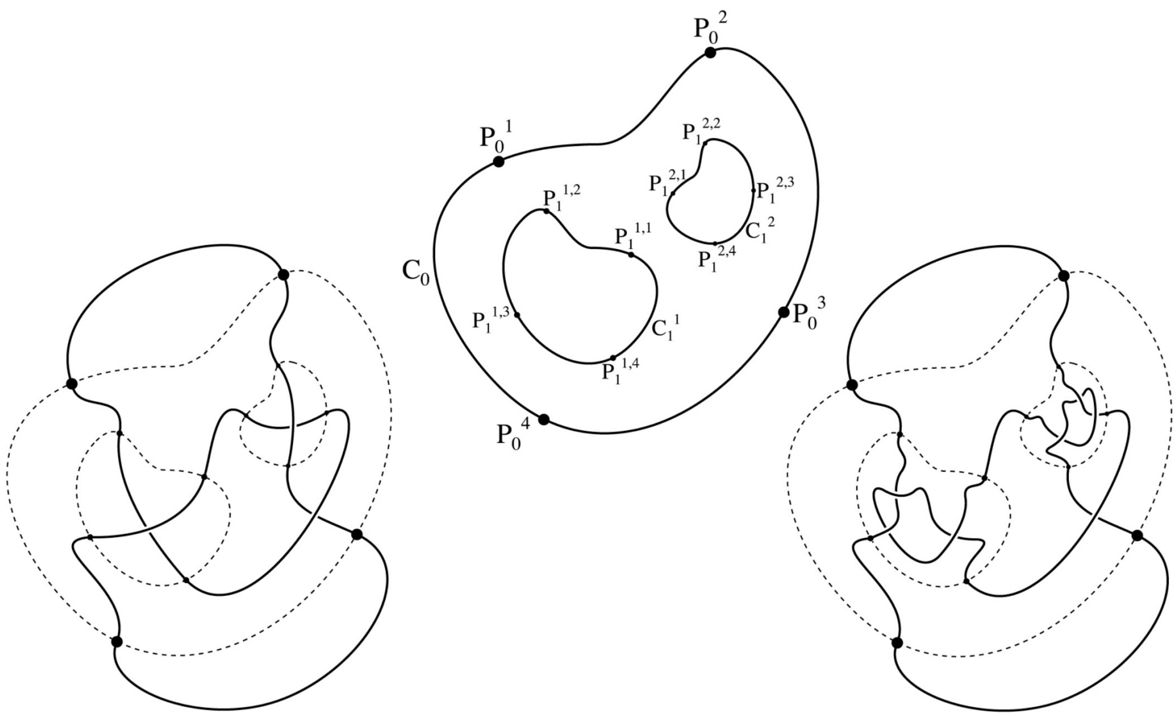

***Figure 1**: An illustration of the method for creating an iterated knot, showing the starting curves and points at top, the starting knot interior plus exterior connections at bottom left, and the knot after a first iteration at bottom right.*

An example of an iterated knot designed according to the method outlined above is shown in Figure 2. The starting curve is a square, and there are two smaller curves, both of which are scaled by 1/3 and neither rotated nor reflected. There are four points on each square at which strands connect, located at midpoints of the edges. The starting knot, shown at top left in Figure 2, has 7 crossings. There is one crossing that is exterior to the largest square, a single crossing within each smaller square, and four crossings in the space between the large and smaller squares. After one iteration, shown at top center in Figure 2, the knot has 17 crossings. The second and third iterations are shown at top right and bottom in Figure 2. The knot is sufficiently complex and the smallest features so small that additional iterations are not shown here. The number of crossings for the second through fifth iterations is 37, 77, 157, and 317. Note that the increase in the number of crossings from one iteration to the next is 10, 20, 40, 80, 160, … The number of crossings N in the jth knot (where $j = 1$ corresponds to the starting knot) is given by the following expression:

$$N(j) = 7 + 10 \sum_{i=0}^{j-1} 2^i = 7 + 10\,(2^j - 1)$$

A general formula can be written for the number of crossings in an iterated knot of this sort. If there are k smaller curves C_1 within C_0, l crossings outside C_0, m crossings in the volume between C_0 and the C_1, and n crossings within each C_1, then the number of crossings in the starting (first) knot is $l + m + kn$. In the iterated (second) knot, the number of crossings is $l + m + k(m + kn)$. In the jth knot, the number of crossings is $N(j) = l + (m + km + k^2m + \ldots + k^{(j-1)}m) + k^jn$. The geometric series in parentheses can be summed, giving for the number of crossings in the jth knot $N(j) = l + m(1 - k^j)/(1 - k) + k^jn$.

3. Design Considerations

Keeping some design guidelines in mind will in general lead to more esthetically pleasing iterated knots. Four general guidelines follow.

A. Keep it simple.

Knots become complex and confusing very quickly. As a result, it is usually a good idea to use a simple starting curve, such as a circle or a polygon possessing symmetry and not too many sides. In addition, keeping the number of curves k small (≤ 5) will also keep things simple. Using the same scaling factor for all of the C_1 will also simplify the construction, but it may be desirable to vary the scaling factor to achieve a particular visual effect.

B. Having some sort of overall symmetry will in general be pleasing to the eye.

Both rotational symmetry and mirror symmetry (one or multiple lines) work well. Note that while the strand positions may obey some symmetry, the signs of the crossings may be reversed from one region of the knot to another and therefore not obey that symmetry.

C. Keep the scaling factor s as close to unity as possible, and the number of iterations small.

This will allow the eye to take in a few iterations at once. After three iterations, the smallest regions will have features only $1/s^3$ as large (linearly) as the largest similar regions. If a scaling factor $s = 1/3$ is used, for example, features in the smallest regions will be 1/27 as large as the similar features in the largest regions.

D. A style should be selected for the finished knot.

One possibility is smooth curves that call to mind rope knots in the real world, as in the knot of Figure 2. It is also possible to use some sharply angled features in otherwise curved strands; this is often employed in Celtic knot designs and was employed in the knot of Figure 3 below. Relatively wide strands are often used in Celtic designs, as in the final knot of Figure 3. Crosses are a design element often found in Celtic knots, and these were used in the knot of Figure 4 below. Islamic style can also be intentionally used to guide the shape of strands, in which case the strands will mostly consist of straight lines and openings in the knot could include characteristic shapes like star polygons. More generally, styles could include bold and forceful, playful (see Figure 5), delicate, etc. Effectively employing a particular style generally requires the reshaping of strands after construction of the basic iterated knot.

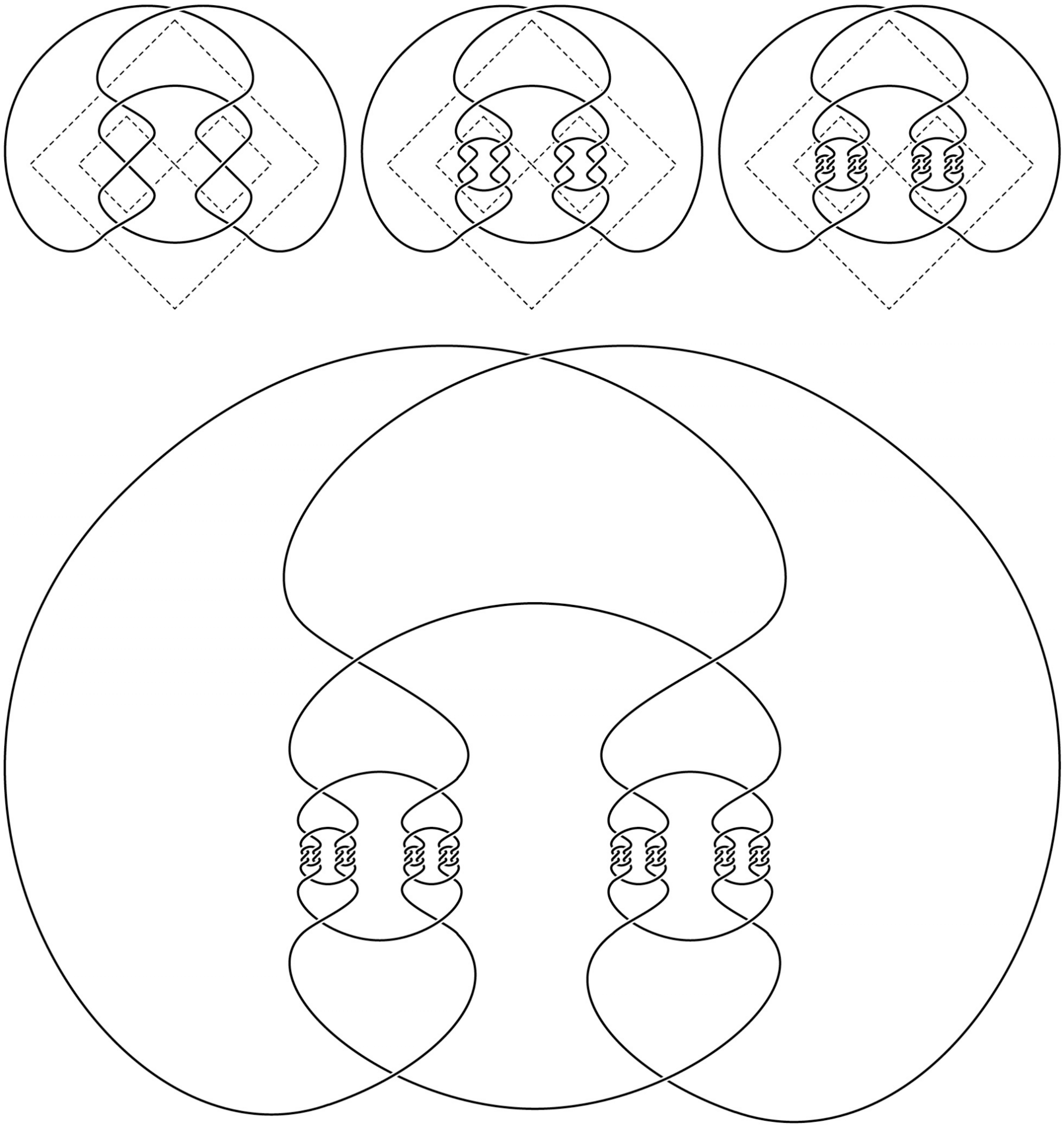

Figure 2: *A simple example of a knot iterated using the method described above. The starting knot and first three iterations are shown. The dashed lines indicated the starting curves described in the method.*

4. Further Examples

Three additional iterated knots designed using the method described above are shown here. The knot of Figure 3 employs a rectangle as the starting curve. Two smaller rectangles are arranged within it, with a rectangular space of the same size and shape between them. The smaller rectangles are rotated 90° with respect to the starting rectangle, and the scaling factor for this arrangement is easily calculated to be $1/\sqrt{3} \approx 0.577$. The starting knot, shown at top left in Figure 3, has three crossings. Since there is only one knot with three crossings, it must be a trefoil knot. The first and second iterations are shown next to it, and the knot after five iterations is shown at the bottom of Figure 3, with gray shading. The number of crossings for the starting knot and first four iterations are 3, 7, 15, 31, and 63, and the increase in the number of crossings with each successive iteration is 4, 8, 16, 32, 64, …

The knot of Figure 4 employs a square as the starting curve, with five smaller squares embedded within it. This was derived from a fractal tiling created *via* dissection of a polyomino that leads to a scaling factor of $1/\sqrt{17} \approx 0.243$ and a rotation angle of $\tan^{-1}(1/4) \approx 14°$ for each of the smaller squares [4]. The starting knot (near top center in Figure 4) has nine crossings, five of which are within the five smaller curves. It has two-fold rotational symmetry and would have four-fold symmetry if not for the exterior strands. Note that some artistic license was taken in connecting points on the smaller squares, with pointed loops that extend well outside the large square. The first iteration is show at upper right in Figure 4, and the second iteration is blown up to overfill the page, in order to make the smallest features clear. In this case, due to the rather severe scaling factor, two iterations already leads to small features. The quintupling of features with successive iterations leads to a rapid increase in the number of crossings, from 9 for the starting knot to 49 for the first iteration to 249 for the second iteration. A third iteration would have 1,249 crossings! The increase in the number of crossings with each successive iteration is 40, 200, 1000, …

The knot of Figure 5 employs a hexagon as the starting curve, with three smaller hexagons embedded within it. The scaling factor is $1/3 \approx 0.333$, and the smaller knotted regions are rotated 60° with each iteration. The strands were given arrow-like shapes in order to create a whimsical design that seems to possess a lot of energy. The starting knot has 15 crossings, and the first two iterations have 30 and 75 crossings. The increase in the number of crossings with each successive iteration is 15, 45, 135, …

5. Conclusions

We have presented a new method for designing iterated knots. In contrast to the previously described method, in which holes in a starting knot are replaced with scaled down copies of the knot, holes in the starting structure in the current method are filled with crossing strands. Several examples have been presented. These illustrate the greater degree of design flexibility in this method, which allows the creation of more stylized and esthetically pleasing iterated knots.

References

[1] Carlo Séquin. *Tangled Knots*, presented at the Art+Math=X conference in Boulder, Colorado, June, 2005. Paper available at http://www.cs.berkeley.edu/~sequin/PAPERS/ArtMath05_TangledKnots.pdf, and PowerPoint presentation at http://www.cs.berkeley.edu/~sequin/TALKS/AM05_TangledKnots.ppt.

[2] See, for example, Colin C. Adams, *The Knot Book,* The American Mathematical Society, Providence, RI, 2004.

[3] Robert Fathauer, *Fractal Knots Created by Iterative Substitution,* Bridges Donostia Conference Proceedings, edited by Reza Sarhangi and Javier Barrallo, pp. 335-342, 2007.
[4] Robert Fathauer, *Fractal Tilings Based on Dissections of Polyominoes,* Bridges London Conference Proceedings, edited by Reza Sarhangi and John Sharp, pp. 293-300, 2006.

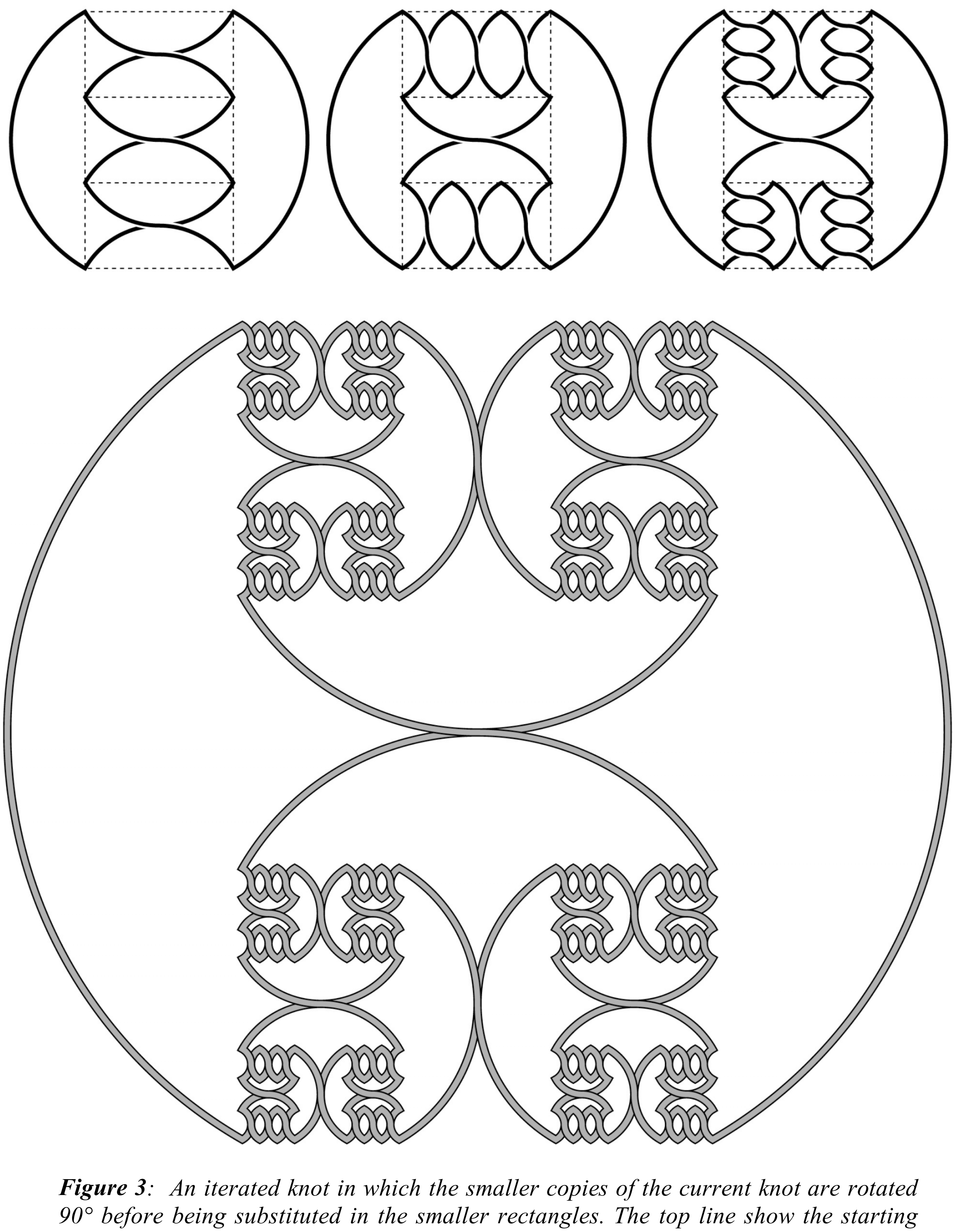

***Figure 3**: An iterated knot in which the smaller copies of the current knot are rotated 90° before being substituted in the smaller rectangles. The top line show the starting knot and first two iterations, while the bottom figure shows the knot after five iterations.*

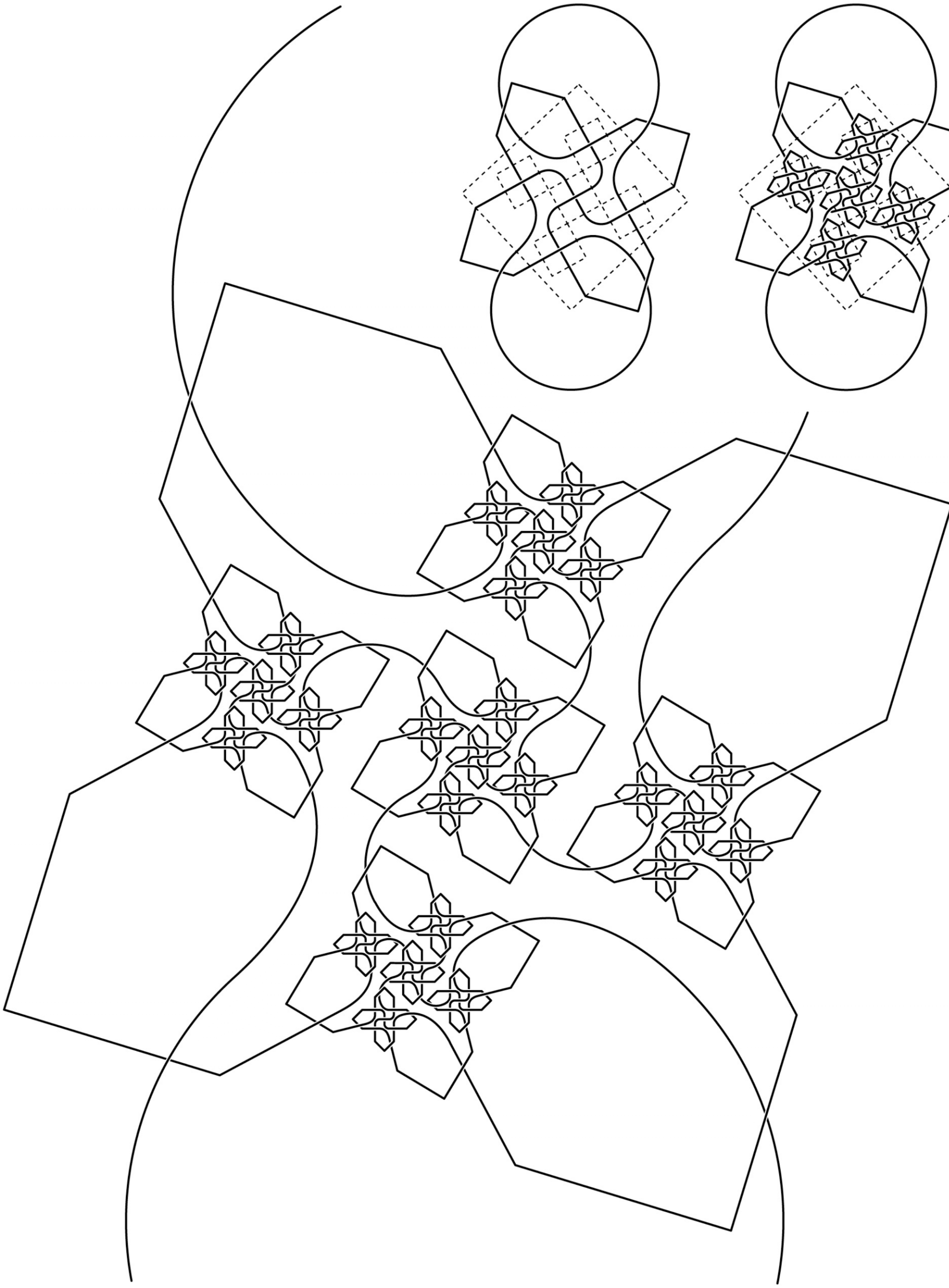

Figure 4: *An iterated knot in which five smaller copies appear with each successive iteration. The starting knot is show at top center, with the first iteration next to it. The knot after two iterations is blown up, with the outer loops cut off, in order to better shown the smallest features.*

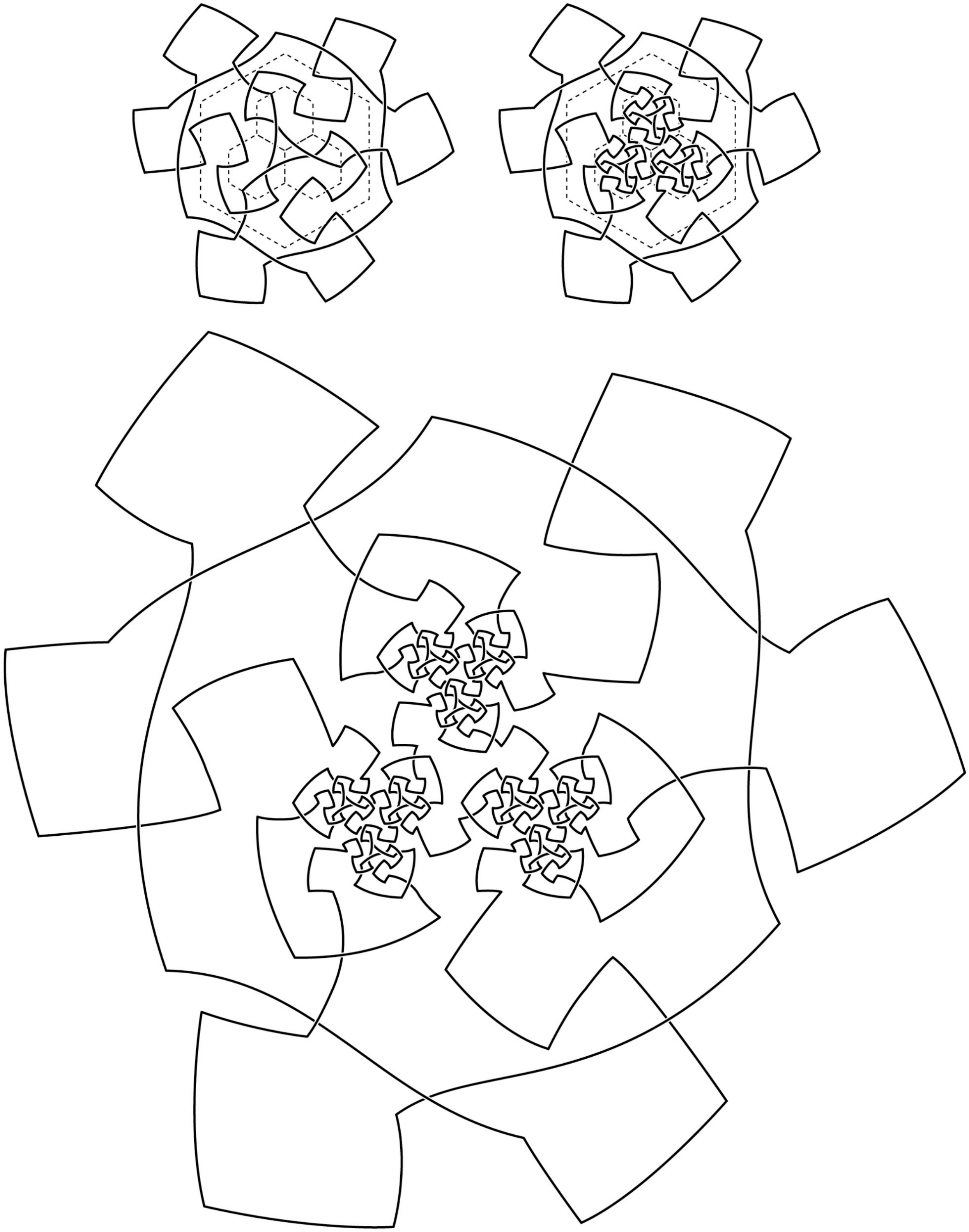

***Figure 5**: An iterated knot with three-fold rotational symmetry. Arrow-like strands were employed to give the knot a vibrant and whimsical character.*