Real Tornado

Akio Hizume, Yoshikazu Yamagishi
Department of Applied Mathematics and Informatics
Ryukoku University
Seta, Otsu, Shiga, Japan

E-mail: akio@starcage.org
yg@rins.ryukoku.ac.jp

Abstract

The continued fraction expansion of a real number \( R > 0 \) generates a family of spiral triangular patterns, called “tornadoes.” Each tornado consists of similar triangles, any two of which are non-congruent.

Basic Operation

Let \( R > 0 \) and \( 0 < s < 1 \). In the plane, the sequence of points \( V(j) = (s^j \cos 2\pi jR, s^j \sin 2\pi jR) \) for \( j = 0,1,\cdots \), which we call the ‘vertices’, naturally converges to the origin. Fix an integer \( k > 0 \), which is called the ‘modulo’ or the ‘step size’, and join the vertex \( V(j) \) with \( V(j+k) \) by the line segment \( V(j)V(j+k) \) for \( j \geq 0 \).

Fibonacci Tornado

The Fibonacci numbers \( f_n \) are defined by \( f_1 = f_2 = 1 \) and \( f_n = f_{n-2} + f_{n-1} \), \( n > 2 \). In the previous paper [2], we showed that if \( k = f_{n-1} \) and \( R = \tau \), where \( \tau = (1+\sqrt{5})/2 \) is the golden ratio, there exists a \( 0 < s < 1 \) such that the vertex \( V(j+f_{n+2}) \) lands on the line segment \( V(j+f_{n+1})V(j+f_{n}) \) for each \( j \geq 0 \). By the Basic Operation above, we obtain the spiral pattern of similar triangles as shown in Figure 1 ( \( k = 2 \) ), which is called a “tornado”. As \( k \) gets larger, we could see that the tornado comes out like a blooming flower, while the argument \( jR \) of each vertex \( V(j) \) remains unchanged.

Remark that the well-known spirals as in Figure 2 are different from our tornadoes because they have congruent triangles.

Figure 1: Fibonacci Tornado. [\tau, 3, 5]
A generic real number $R$ also generates a family of tornadoes. As is well-known (see [1]), the continued fraction expansion of $R$ as in the Figure 3 is defined by $R = C_0 + \frac{p_0}{q_0}$, $0 \leq \varepsilon_0 < 1$, and $1 / \varepsilon_n = C_{n+1} + \varepsilon_{n+1}$, $0 \leq \varepsilon_{n+1} < 1$ for $n \geq 0$, where $C_n$ are called the partial denominators. If $R$ is rational, it is related to the Euclidean algorithm and stops when $\varepsilon_n = 0$. The $n$-th convergent $p_n / q_n$ is defined by $p_0 = C_0$, $q_0 = 1$, $p_1 = C_1 p_0 + 1$, $q_1 = C_1$, and $p_{n+1} = C_{n+1} p_n + p_{n-1}$, $q_{n+1} = C_{n+1} q_n + q_{n-1}$ for $n > 0$. It is known that $p_n / q_n$ are the best approximations of $R$, where

$$\frac{P_0}{q_0} < \frac{P_2}{q_2} < \cdots < R < \cdots < \frac{P_3}{q_3} < \frac{P_n}{q_n}, \quad \text{and} \quad \left| \frac{P_n}{q_n} - R \right| \leq \frac{1}{q_n q_{n+1}} \quad \text{for} \quad n \geq 0.$$ 

For example, the convergents of $R = \sqrt{5}$ are $1/1$, $2/1$, $5/3$, $7/4$, $19/11$, $26/15$, $71/41$, .... The denominators $q_n$ and $q_{n+1}$ are coprime.

Choose any pair of consecutive convergents $p_n / q_n$ and $p_{n+1} / q_{n+1}$, and denote by $q = q_n$ and $q' = q_{n+1}$. Define the step size by $k = q' - q$. Then there exists a unique $0 < s < 1$ such that under the Basic Operation the vertex $V(j + q + q')$ lands on the segment $V(j + q)V(j + q')$ and we obtain a spiral pattern named as the tornado $[R, q, q']$, consisting of similar triangles $T_j = \Delta V(j)V(j + q)V(j + q')$ for $j \geq 0$. Figure 4 presents the tornadoes $[R, q, q'] = [\sqrt{3}, 3, 4]$ and $[\sqrt{3}, 4, 11]$.

The basic idea of the Real Tornado was originally published in Japanese in [3]. Here we show how to find a $0 < s < 1$. Denote the length of the three edges of $T_j$ by

$$a(j) = \left| V(j + q)V(j + q') \right|,$$

$$b(j) = \left| V(j)V(j + q) \right|,$$

$$c(j) = \left| V(j)V(j + q') \right|.$$
By Figure 5 we can see that
\[
a(j) = |V(j + q)V(j + q')| \\
= |V(j + q)V(j + q + q')| + |V(j + q + q')V(j + q')| \\
= s^q|V(j)V(j + q')| + s^q|V(j)V(j + q)| \\
= s^q c(j) + s^{q+k} b(j).
\]
The three angles of \( T_j \) are
\[
\phi = 2\pi R q' = 2\pi R (q + k), \quad \delta = -2\pi R q, \quad \text{and} \quad \theta = 2\pi R k
\]
or
\[
\phi = -2\pi R q' = -2\pi R (q + k), \quad \delta = 2\pi R q, \quad \text{and} \quad \theta = -2\pi R k,
\]
where the signs are chosen to satisfy that \( \sin \phi, \sin \delta \) and \( \sin \theta \) are all positive. The law of sines is expressed by
\[
\frac{a(j)}{\sin \theta} = \frac{b(j)}{\sin \delta} = \frac{c(j)}{\sin \phi},
\]
and we obtain the equation
\[
s^q \sin(2\pi R q) - s^q \sin(2\pi R q') + \sin(2\pi R k) = 0.
\]
It is easy to see that this equation has a unique solution \( 0 < s < 1 \).

**Additional Results**

Conversely, we can also prove that any possible tornado \([R, q, q']\) with \( q, q' \) positive is related to the continued fraction expansion of \( R \).

Theorem: Let \( R \) be a real number and \( q, q' \) positive integers. There exists a tornado \([R, q, q']\) if and only if \( R \) has a convergent \( \frac{p}{q} \) and an (intermediate) convergent \( \frac{c p_n + p_{n-1}}{c q_n + q_{n-1}} \), where we denote
\[
q_n, p_n, q_{n+1}, p_{n+1}, c_{n-1} 
\]
by \( p = p_n, q = q_n, p' = c p_n + p_{n+1} \) and \( q' = c q_n + q_{n+1} \), such that

1. \( R \) is distinct from \( \frac{p}{q} \) and \( \frac{p'}{q'} \), that is, \( \frac{p}{q} < R < \frac{p'}{q'} \) or \( \frac{p'}{q'} < R < \frac{p}{q} \), and

2. \(|\{qR\} - \{q'R\}| > 1/2 \), where \( 0 \leq \{x\} = x - \lfloor x \rfloor < 1 \) denotes the fractional part.

See [4] for the proof and further discussions. Note that the golden ratio \( \tau \) is a special irrational number which has no intermediate convergents.

**Acknowledgements**

The authors would like to thank the reviewers for their helpful comments and suggestions. They suggested to consider the equation \( z^{q+k} = \alpha z^k + (1-\alpha) \) with \( 0 < \alpha < 1 \) given, where \( q \) and \( k \) are relatively prime. By experiments, they claim that the tornado \([R, q, q + k]\) is obtained by using the root
\[ z = se^{2\pi iR} \neq 1 \] of the largest magnitude. Note that in our setting above, the ratio \( \alpha \) tends to 0 or 1 as \( R \) approaches to \( p/q \) or \( p'/q' \) respectively.

References


\[ \begin{align*}
\text{Figure 6}: & \text{ Real Tornado Samples.}
\end{align*} \]