Aesthetically Pleasing Azulejo Patterns

Russell Jay Hendel
Mathematics Department, Room 312
Towson University
7800 York Road
Towson, MD, 21252, USA
E-mail: RHendel@Towson.Edu

Abstract

In [1], Fernandez outlined an algebraic approach to the construction of Azulejo patterns. An open problem listed in this paper is "to describe the geometric features of these designs that account for their aesthetic properties." This paper points to a direction to answer this problem. We identify *aesthetic pleasantness* with *aesthetic challenge* and present two definitions of *aesthetic challenge*: (1) *Aesthetic challenge* can be partially defined and quantified by the number of English phrases needed to describe an Azulejo pattern; (2) We can also define *aesthetic challenge* as equal to the number of primitive figures – isolated points and lines – in the *fundamental octant* of the Azulejo pattern. We show that these two definitions correlate and that they provide an alternative to Fernandez's purely algebraic approach for construction of Azulejo patterns. Several examples are presented showing that both approaches – the algebraic approach of Fernandez and the *aesthetic challenge* approach of the author - yield aesthetically pleasing designs. Future directions of research on the definition of *aesthetic challenge* are presented.

1. Azulejo Patterns

Azulejos are the traditional designs used by Spanish artisans for many generations. They typically have a blue and white design painted on a ceramic tile. Federico Fernandez, a professional architect, described the mathematical properties of these varied and beautiful designs in [1]. Some examples of Azulejo and related patterns are presented below in Figure 1 which the reader is encouraged to review now.

Intuitively, most people would consider the aesthetic pleasantness of the patterns in Figure 1 to increase as one goes from the two patterns in the leftmost column to the two patterns in the rightmost column. Fernandez's goal, in his paper, was to find an algorithm that would *exclusively* produce patterns considered aesthetically pleasing by most people. The Fernandez algorithm does not, for example, produce the two patterns in the leftmost column of Figure 1. There are however, two areas for improvement in the Fernandez algorithm: (1) The Fernandez algorithm may occasionally produce patterns that are not considered aesthetically pleasing; the algorithm user must simply discard those patterns and produce new ones. (2) The Fernandez algorithm simply produces patterns without in any way distinguishing those patterns that are *more* aesthetically pleasing than others. For example, the Fernandez algorithm does not distinguish between the aesthetic appeal of the patterns in the middle and rightmost columns of Figure 1.

In an attempt to remedy these problems, in Sections 3 and 4 we present two definitions of *aesthetic pleasantness* or *aesthetic challenge*, both definitions involving numerically quantifiable entities. We show that these two definitions highly correleate. Using these two definitions we proceed, in Section 4, to present an alternative to the Fernandez algorithm for producing Azulejo patterns. The patterns produced by the algorithm presented in this paper seem comparable in aesthetic pleasantness to the patterns produced by the Fernandez algorithm. Nevertheless, as with the Fernandez algorithm, the algorithms presented in this paper occasionally produce patterns that most people would not consider pleasing.

In this introductory section, we briefly sketch some important features of the Fernandez algorithm and indicate major areas where the approach presented in this paper differs.

Fernandez *starts* Azulejo construction by computing the orbits of certain matrices acting on the integer lattice **Z x Z** and then extends the resulting sets by closing them under the actions of a subgroup of the dihedral group, $\mathbf{D}_4$. Fernandez's method also allows elimination of the "simplistic" patterns presented in the leftmost column of Figure 1. As already pointed out the resulting Fernandez algorithm occasionally produces patterns that are not considered aesthetically pleasing and which must be discarded and replaced

In a closing section of his paper Fernandez points to certain constructs, the *fundamental octant* and the *fundamental N-square,* which he says may point the way to a solution of the problem of defining *aesthetic pleasantness.*

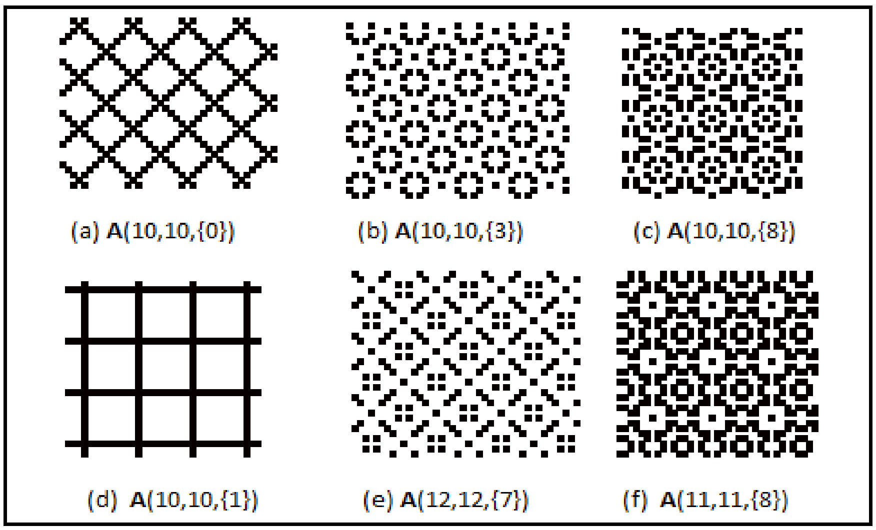

Figure 1: *Six patterns with varying levels of aesthetic challenge. Figures 1(b),(c),(e) and (f) follow the method in the Fernandez paper. Figures 1(a) and (d) are rejected as Azulejos by the Fernandez method but are brought here for purposes of contrast. More detail is presented in the text.*

The approach presented in this paper differs in two ways from the approach of Fernandez. First, in this paper we reverse the sequence in the Fernandez algorithm: Our approach to construction of Azulejo patterns, presented in Section 2, *starts* with the construction of a fundamental *octant*, a subset of $\mathbf{Z_n} \textbf{ x } \mathbf{Z_n}$, and then proceeds to extend this set of points by closing them under the actions of the dihedral group, $\mathbf{D_4}$. This gives us a *fundamental* **n x n** square from which we can build up the entire Azulejo pattern by a process of translation. The second and major innovation introduced in this paper is that we do not use algebraic methods to construct the *fundamental octant.* Instead, we use *complexity,* a numerical quantity, defined in Sections 3 and 4, that seems to correlate with *aesthetic pleasantness*.

2. The Azulejo Algorithm

In this section we present an algorithm for construction of Azulejo patterns that is equivalent to the Fernandez algorithm. We also indicate which algorithm steps are modified by our approach. To facilitate the readability of this section which is addressed both to mathematicians and to a less technical audience we first describe the algorithm visually and then present the algorithm formally.

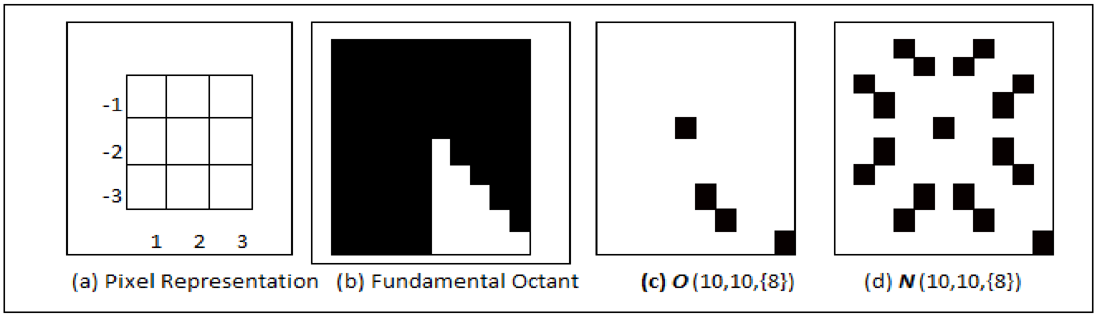

Figure 2: *Fundamental octant* (c) and *fundamental N-square* (d) *for the Azulejo presented in Figure 1(b).*

Informal visual description. The Fernandez algorithm has five steps. They will be illustrated using the Azulejo presented in Figure 1(b). Figures supporting the algorithm description are presented in Figure 2.

1. Pixelation: It is important to emphasize that the Fernandez algorithm begins by perceiving the plane as represented by a grid of pixels. Each pixel has a unique address given by the Cartesian coordinates of its lower right corner. This is illustrated in Figure 2(a). As is traditional, we identify each pixel with its address. For example the pixel in the lower left corner in Figure 2(a) is identified with the "address" (1,-3). Fernandez introduced pixelation since it powerfully allows the description of complex figures and patterns using simply computed arithmetic functions. For computational reasons we use a square with positive x coordinates and negative y coordinates.
2. The fundamental octant: To begin Azulejo construction one starts with an n x n square of pixels as illustrated in Figure 2(a). Here, n is an arbitrary integer selected by the algorithm user. We then select the pixels in the lower half of the lower right quadrant, illustrated in Figure 2(b). We call this lower half of the lower right quadrant the *fundamental octant*.
3. The octant pattern: The Fernandez algorithm requires filling-in, or coloring black, certain pixels in the *fundamental octant*. The addresses of the pixels to be colored black are given by arithmetic functions. Figure 2(c) illustrates one such octant pattern. Details are provided in the next subsection.
4. The *N*-square pattern: If the reader carefully studies the relation between Figures 2(c) and (d) (s)he will see that the octant pattern in Figure 2(c) is simply reflected vertically, horizontally and diagonally in Figure 2(d). More precisely, Figure 2(c) consists of the pixel in the lower right corner, with an additional three-pixel pattern, while Figure 2(d) consists of the pixel in the lower right corner and an eight-fold repetition of the three-pixel pattern. This eight-fold repetition is accomplished by applying all combinations of horizontal, vertical and diagonal reflections. A formal description of the symmetries that transform Figure 2(c) into Figure 2(d) is given in the next subsection.
5. The Azulejo pattern: If the reader carefully studies the relation between Figure 1(b) and Figure 2(d) (s)he will see that the Azulejo presented in Figure 1(b) is simply a "translation" of Figure 2(d) throughout the plane.

Formal Description. The formal description of the five steps listed in the last subsection are as follows:

1. Pixelation: We use a traditional Cartesian system. The pixel associated with address (x,y) is the unique pixel whose lower right coordinates are (x,y) – e.g. (1,-3) is the lower left corner of Figure 2(a)
2. The fundamental octant: To formalize the algorithm we suppose that n is a positive integer with positive divisor d and k is an integer with $0 \leq k \leq d - 1$. The *fundamental octant* is defined as the collection of pixels whose addresses (x,y) satisfy the *fundamental octant inequality*, $n/2 \leq x,-y \leq n$. Throughout this section we illustrate with $n = 10$, $d = 10$ and $k = 3$.
3. The octant pattern:
   a. To describe the pixels that will be colored black in the *fundamental octant*, Fernandez begins with the function $x(d,0) + y(k, n/d)$ where x and y are arbitrary integers. For example, when

$x = 6$ and $y = 7$ this function equals $6(10,0) + 7(3,10/10) = (81,7)$.

b. We next introduce the function [ ] defined as follows: if x and y are integers, then we define $[(x,y)] = (a,b)$, with $1 \leq a,-b \leq n$ and $x \equiv a$, $y \equiv b$ (mod n). For example, $[(81,7)] = (1,-3)$. Heuristically, [ ] maps each pixel to its translate in the **[1,n] x [-1,-n]** square.

c. We need a function to take the pixel (1,-3) to its "related pixel" in the *fundamental octant*. Formally, for $1 \leq x,-y \leq n$, we define a function by $<(x,y)> = (a,b)$, with $n/2 \leq a,-b \leq n$ and (a,b) is a member of $\{[(\pm x, \pm y)], [(\pm y, \pm x)]\}$; e.g., $<(1,-3)> = (9,-7)$.

d. Summary: Fernandez, by applying the functions [ ] and < > to the function $x(d,0) + y(k, n/d)$, where x and y are arbitrary integers, associates each x and y with a pixel in the *fundamental octant* which is to be colored black.

4. The *N*-square pattern: Finally, we need an arithmetic method to take each pixel in the *fundamental octant,* and give all "related pixels" in the *fundamental N-square*. Formally, for $n/2 \leq x,-y \leq n$ we define $<<(x,y)>> = \{(a,b)\colon <(a,b)> = <(x,y)> \text{ and } 1 \leq a,-b \leq n\}$. For example, $<<(1,-3)>> = \{(1,-3), (1,-7), (9,-3), (9,-7), (7,-9), (7,-1), (3,-1), (3,-9)\}$. The mathematical reader will recognize << >> as giving the *orbit* of a pixel modulo n under the actions of the dihedral group, $\mathbf{D_4}$. Heuristically, the dihedral group takes the pattern in the *fundamental octant* and reflects it in all 8 combinations of vertical, horizontal and diagonal symmetry.

5. The Azulejo pattern: Let ***O*(*n,d,k*)** denote the set of pixels in the *fundamental octant*. We define the *fundamental N-square*, ***N*(*n,d,k*)** = <<***O*(*n,d,k*)**>>. Here the function << >> applied to a set is simply the union of << >> applied to all members of that set. Heuristically, ***N*** is the orbit of ***O*** under the actions, modulo n, of the dihedral group, $\mathbf{D_4}$, and consequently ***N*** lies in the rectangle of pixels **[1,*n*] x [-1,-*n*]**). Finally, we define, ***A*(*n,d,k*) = ∪ {*N*(*n,d,k*) + (*x,y*)): *x* and *y varying over all integers,*}** (that is, ***A*** is the union of all integral translates of the ***N*** square)

To increase richness and variety of patterns Fernandez allowed construction of Azulejos based on a set of integers K. Here, ***O*(*n,d,K*)** is simply the set-theoretic union of the ***O*(*n,d,k*)** as k varies over members of K. We then define ***N*(*n,d,K*)** and ***A*(*n,d,K*)** in the obvious ways. Examples are given in later sections.

The author's algorithm: The author's algorithm differs from the Fernandez algorithm only in step 3: Whereas Fernandez used the function $x(d,0) + y(k, n/d)$ to generate the pixels in the *fundamental octant* pattern, the author allows a random selection of pixels to be colored black provided the selected set of pixels has sufficient *complexity*. *Complexity* is defined in the next two sections.

3. Aesthetic Pleasantness

In this and the next section we outline our approach to the definition of *aesthetically pleasing*. We first identify *aesthetic pleasantness* with *aesthetic challenge.* In this section we explore the informal heuristic that the *aesthetic challenge* of an Azulejo pattern can be numerically quantified by *the number of English phrases needed to describe it*. We illustrate this approach using the three columns of Azulejo patterns presented in Figure 1:

- The leftmost column has aesthetic challenge 1: This corresponds to the fact that we would describe Figure 1(d) as a *lattice of squares* and Figure 1(a) as a *lattice of diamonds.* The numerical value of *aesthetic challenge* equal to one comes from the single underlined term needed to describe the figure. (Fernandez ruled out these patterns as Azulejo patterns by using algebraic criteria. However our approach will be to allow all patterns, classify them, and then only use the more *complex* ones.)
- The middle column has aesthetic challenge 2: This corresponds to the fact that we would describe Figure 1(b) as an *alternating lattice of circles and points,* or as an *alternating lattice of octagons and points,* and we would describe Figure 1(e) as *a lattice of diamonds with inscribed squares.*

The numerical value of *aesthetic challenge* equal to two comes from the two underlined terms needed to describe the figure. Notice that these descriptions are not complete. For example, the diamonds in Figure 1(e) use punctured lines for their sides. However, the basic bird's-eyeview description of the figure is *diamonds with inscribed squares.* As indicated above, this method of measuring *aesthetic pleasantness* by counting phrases, is informal.

- The rightmost column has aesthetic challenge 3+: To describe either pattern in the rightmost column we need at least three descriptors, possibly more. For example, Figure 1(c) has the following subthemes: (1) octagons of points with a point in the center, (2) four sets of a pair of solid rectangles surrounding a blank square and (3) two opposing sets of solid rectangle pairs surrounding a point. There are of course other ways of describing the Azulejo pattern but they all involve a reference to at least three subthemes. Similarly Figure 1(f) may be described using the following subthemes: (1) A white square with a central dot, (2) two opposing solid rectangles with a white slit separating them and white center, (3) a white dot, inscribed in a black square, inscribed in a rounded white square with ornaments. Again, other descriptions are possible but they all would use at least three basic subthemes. Also note that the above descriptions of Figure 1(c) and Figure 1(f) are informal in the precise sense that an outside person could not *reproduce* the Azulejo pattern from the three subthemes because there is *further richness* to the pattern.

Frequently, in a mathematics paper, definitions are presented as is without further defense. However, in this paper we are using a definition to capture a human concept - *aesthetic pleasantness.* We therefore review the heuristic arguments for *number of English phrases* capturing *aesthetic pleasantness*:

The basic psychological assumption used is that *the challenge to the viewer to identify patterns in a figure creates a sense of accomplishment, satisfaction, and aesthetic pleasantness*. The viewer of Figures 1(a), (b), (d), and (e) can instantly recognize and understand these patterns; there is no further challenge; hence these patterns are not aesthetically challenging. By contrast, Figures 1(c) and (f) are more challenging, since more sub-patterns must be identified. Note particularly that a random array of dots would not be as challenging. The important point to emphasize is that there is *challenge coupled with partial accomplishment.* Consequently, Figures 1(c) and (f) are *very* challenging because the viewer trying to discern the multiple underlying patterns only achieves a partial success by identifying several various subthemes which however only cover a significant part of the pattern.

For purposes of this paper we will suffice with (arbitrarily) categorizing *challenge levels* as equal to 1, 2 and 3+. In the next section, we continue our attempt to numerically quantify *aesthetic challenge*.

4. Fundamental Octant Complexity

As explained in the last section, Figure 1(b) requires two English keywords to describe the pattern. Heuristically, we would like to argue that the *fundamental octant* for this figure, exhibited in Figure 2(c), is overly simplistic, containing one line segment and 2 pixels; this octant simplicity gives rise to the simplicity of Figure 1(b). To facilitate formalizing this intuition of *octant complexity* we first present a simple lemma describing the relation between the *fundamental octant* and the full Azulejo pattern.

Lemma 1**:** Let $\boldsymbol{O}$ be the fundamental octant with fundamental square $\boldsymbol{N}$, of an Azulejo pattern, $\boldsymbol{A}(\boldsymbol{n},\boldsymbol{d},\boldsymbol{K})$.

(a) A singleton pixel, $(n,-n)$ in the set $\boldsymbol{O}$ gives rise to one pixel in $\boldsymbol{N}$. Similarly if n is even then $(n/2, -n/2)$ in the set $\boldsymbol{O}$ gives rise to one pixel in $\boldsymbol{N}$.
(b) A singleton pixel $(m,-m)$ in the set $\boldsymbol{O}$, with $m \neq n$, $m \neq n/2$, gives rise to 4 pixels in $\boldsymbol{N}$.
(c) A singleton pixel $(n,-k)$ in the set $\boldsymbol{O}$, with $k \neq n$, $k \neq n/2$ gives rise to 4 pixels in $\boldsymbol{N}$.
(d) A singleton pixel $(j,-k)$ in the set $\boldsymbol{O}$, with $j \neq k$, $j,k \neq n$, $j,k \neq n/2$, gives rise to 8 pixels in $\boldsymbol{N}$.

(e) If n is even then the singleton pixel $(n, -n/2)$ in $\boldsymbol{O}$ gives rise to 2 pixels in N.

Proof: Parts (a) - (e) all follow from the the definition of << >> and the *fundamental octant inequality*. For example <<$(n,-n)$>> = $\{(n,-n)\}$ since $[(\pm n, \pm n)] = (n,-n)$ by our conventions of representatives of $\mathbf{Z_n}$. Similarly for case (d) the << >> function produces the orbit of $(j,-k)$ under the dihedral group, $\mathbf{D_4}$, which has 8 pixels.

We now define, $\boldsymbol{C} = \boldsymbol{C}(\boldsymbol{n,d,K})$, the octant *complexity* level, as follows:

a) For each vertical, horizontal, diagonal and anti-diagonal "line" with two or more pixels in $\boldsymbol{O} - \{(n,-n)\}$ we assign a value of one; (if a pixel in $\boldsymbol{O}$ - $\{(n,-n)\}$ is shared by two such lines then both lines are assigned a value of one); here, by the term "line" we mean *any* collection of filled-in pixels, not necessarily consecutive, whose addresses satisfy a linear equation (note, since we only allow horizontal, vertical and (anti)-diagonal lines the slope must be 0,1,-1, or infinity).
b) For each remaining pixel not on any of the lines enumerated in (a) we assign a value of one.

$\boldsymbol{C}$ is defined as the sum of assigned values.

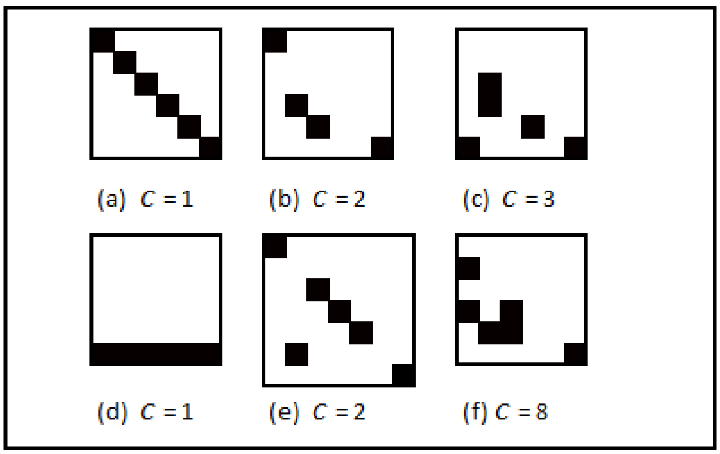

Figure 3: *The fundamental octants and their complexities, for the six Azulejo patterns in Figure 1. The verification of the numerical complexities is left as an exercise for the reader.*

Three points should be noted about our definition of *complexity*: (1) The numerical *complexity, C,* increases as one goes from the leftmost column to the rightmost column in Figure 1. (2) The octant complexity, C, correlates with the *number-of-English-phrases* complexity introduced in the last section. (3) Our definition of complexity surprisingly avoids any mention of symmetry! The Section-2 description of the Fernandez algorithm shows an ingenious separation of a symmetry component (Step 4) from a non-symmetry component (Step 3). That is, the octant pattern (Step 3) *need not* have any symmetry since by applying the dihedral group symmetries (Step 4) the resulting Azulejo will have symmetry.

We are now ready to identify the non-algebraic criteria used in this paper for constructing octants.

Definition: An octant is said to be *minimally aesthetically challenging* if its *octant complexity* is at least 3. An octant is said to be *very aesthetically challenging* if its *octant complexity* is at least 10 (this choice of 10 in the definition is somewhat arbitrary). The corresponding Azulejo pattern is said to be *minimally* or *very aesthetically challenging* if its octant is *minimally* or *very aesthetically challenging*.

As already indicated at the end of Section 2, Fernandez constructed the fundamental octant using certain simple linear arithmetic functions. The position of the author is that a random approach to octant pattern construction is *equally* capable of producing aesthetically pleasing patterns.

5. Aesthetically Pleasing Azulejos Based on Random Octants

The goal of this section is to compare the *aesthetic pleasantness* of Azulejos whose octant construction is based on *complexity* vs. an algebraic definition. The octants and corresponding Azulejos are presented in Figures 4 and 5 respectively which the reader is invited to review now. The reader can readily compare the aesthetic pleasantness of these Azulejos to verify the thesis of this paper that a *complexity* approach to octant construction is a viable alternative to Fernandez's algebraic approach. The following additional points should be made:

Figures 5(a) and 5(d) use Fernandez's algebraic approach. The octants each have $C > 20$. As can be seen the Azulejos are indeed *very aesthetically challenging*. By contrast, Figures 5(b), (c), (e) and (f) use a pure *complexity* approach without an underlying algebraic generation. Figures 4(b), (c), (e) and (f) should be approached as follows:

Figure 4(b) was a first attempt at aesthetic challenge without algebra. The octant has one pixel, one diagonal (left to right upward) and one vertical line. Its complexity is five (there are two extra downward diagonals (left to right). Figure 5(b) shows the corresponding Azulejo. It can be described with three English phrases: (1) a pixel (2) inscribed in a diamond, (3) inscribed in a 4-corner-square. The three English phrases needed to describe the Azulejo clearly correspond to the three connected components of the octant: by lemma 1, the pixel in the lower right corner remains a pixel in the Azulejo, the upward (left to right) diagonal becomes a diamond, and the vertical line becomes a 4-corner-square.

Figure 5(b) is only *minimally aesthetically challenging*. It can be *completely described* with three English phrases. Figures 4(c), (e), and (f) are attempts to remedy this simplicity.

- Figure 4(c) "fixes" Figure 4(b) by adding an extra dot and an extra upward diagonal. The resulting Azulejo now requires five English phrases to describe it: a (1) pixel (2) inscribed inside a diamond (3) inscribed in a 4-pixel square (4) inscribed in a 4-corner square. Each set of four of these four-fold nested patterns forms a square with (5) a diamond in the center. Although this Azulejo has complexity nine it is too transparent in the precise sense that it can be *completely* described by the five phrases.
- Figure 4(e) "fixes" Figure 4(b) by adding a "random" assortment of dots. The resulting Azulejo pattern, Figure 5(e), is more aesthetically challenging in the precise sense that it can't be *completely described.*
- Figure 4(f) goes one step beyond Figure 4(e) by providing a pure random assortment of dots without any lines. Although $\boldsymbol{C} = 5$, this pattern appears more aesthetically challenging than Figure 5(c) whose complexity equals 9. The observation that many people consider Figure 4(f), with $C = 5$, more aesthetically challenging than Figure 4(c), with $C = 9$, strongly suggests that the definitions of *aesthetic challenge* presented in this paper are only useful approximations and that further research into a better definition of aesthetic challenge is needed.

The above remarks suggest the following future directions of research, which were already raised in [1]. (1) Do random dot assortments which miss lines produce aesthetically pleasing Azelujos? – if so, is there a way of numerically quantifying the complexity of random dot patterns? (2) When there are competing lines, diagonals and pixels in an octant which ones will stand out in the resulting Azulejo? (3) What algebraic and graph theoretic properties of octants result in Azulejos that are *completely describable*? (4) Figures 4(e) and 4(f) use *knight* jumps (1 diagonally down and 1 over) to produce the *random* dot configurations; is this method generalizable?

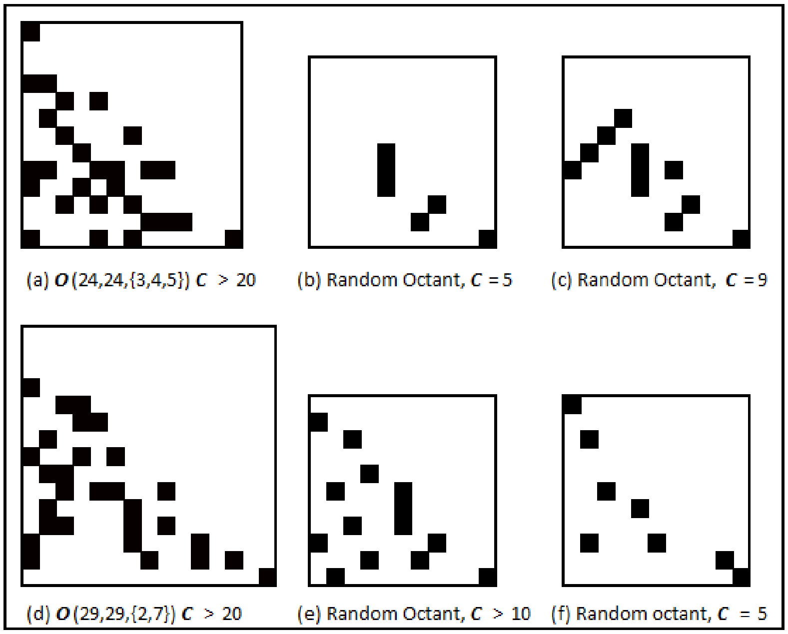

Figure 4: *Comparison of octants based on an algebraic definition (Figures a,d) vs. a complexity approach (Figures b, e, c, f). The corresponding Azulejos are presented in Figure 5.*

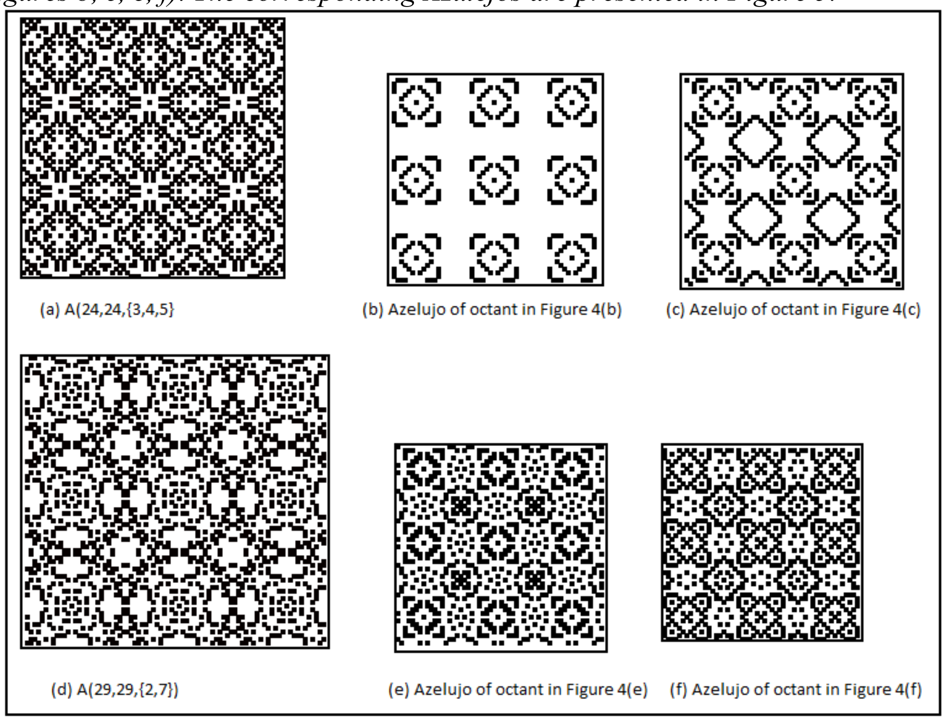

Figure 5: *Azulejo patterns for the fundamental octants of Figure 4.*

References

[1] Federico Fernandez, *A Class of Pleasing Periodic Designs*, The College Math Journal, Vol. **29.1**, pp. 18-26. 1998.