

The Borromean Rings: A Video about the New IMU Logo

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Abstract

This paper describes our video *The Borromean Rings: A new logo for the IMU*, which was premiered at the opening ceremony of the last International Congress. The video explains some of the mathematics behind the logo of the International Mathematical Union, which is based on the tight configuration of the Borromean rings. This configuration has pyritohedral symmetry, so the video includes an exploration of this interesting symmetry group.

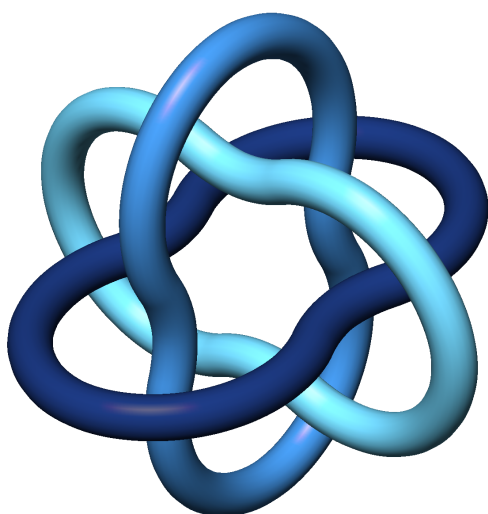


Figure 1: The IMU logo depicts the tight configuration of the Borromean rings. Its symmetry is pyritohedral, as defined in Section 3.

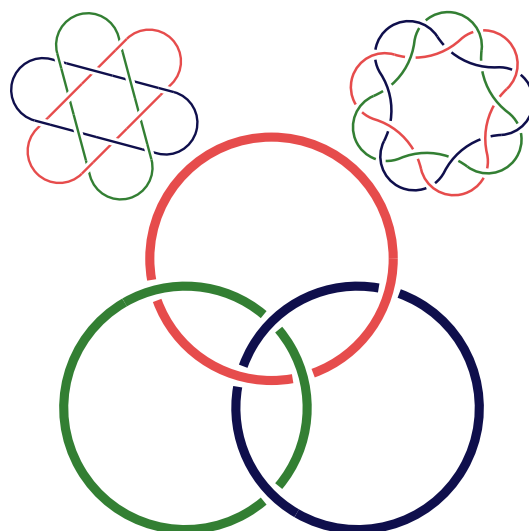


Figure 2: A typical diagram for the Borromean rings uses three round circles, with alternating crossings. In the upper corners are diagrams for two other three-component Brunnian links.

1 The IMU Logo and the Borromean Rings

In 2004, the International Mathematical Union (IMU), which had never had a logo, announced a competition to design one. The winning entry, shown in Figure 1, was designed by one of us (Sullivan) with help from Nancy Wrinkle. It depicts the Borromean rings, not in the usual diagram (Figure 2) but instead in their tight configuration, the shape they have when tied tight in thick rope. This IMU logo was unveiled at the opening ceremony of the International Congress of Mathematicians (ICM 2006) in Madrid. We were invited to produce a short video [10] about some of the mathematics behind the logo; it was shown at the opening and closing ceremonies, and can be viewed at www.isama.org/jms/Videos/imu/.

The Borromean rings form a three-component link. (Mathematicians think of a knot as a simple closed curve in space, and a link as a union of several curves, which may be individually knotted and/or entangled with each other.) Each component of the Borromean rings is unknotted, and each pair of components is unlinked. That is, although the three components taken together are nontrivially linked, if any one of them is removed, the other two can fall apart, as in Figure 3. In general, a *Brunnian link* of n components is one in which removing any single component results in the unlink of $n - 1$ components; see Figure 2.

This Brunnian property has led the Borromean rings to be used—over many centuries, and in many cultures—as a symbol of interconnectedness or of strength through unity. Indeed, their name comes from their use in the crest of the Italian noble family Borromeo. Figure 4 shows some examples from Japan and Italy, as seen in our video; Peter Cromwell’s website [5] has many more examples.

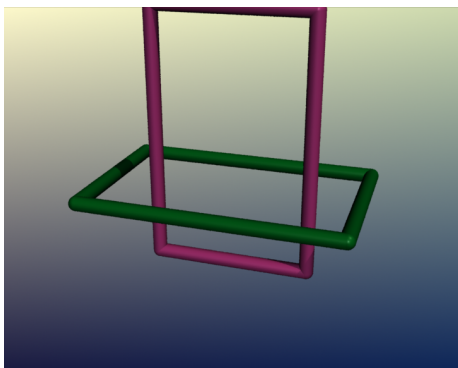


Figure 3: The Borromean rings form a Brunnian link: if any one component is removed, the other two come apart. The rings have thus been used as a symbol of interconnectedness.



Figure 4: Photos of the Borromean rings from the churches Santa Croce (Florence, left, by Sullivan), and San Sigismondo (Cremona, bottom right, by Peter Cromwell), and the Shinto shrine O-Miwa Jinja (near Nara, top right, by Philippe Lavergne).

Most often—especially for such nonmathematical uses—the Borromean rings are drawn in a diagram made from three round circles, as in Figure 2. The three circles overlap as in the Venn diagram showing the various intersections of three sets, and the six crossings are chosen to alternate over/under along each ring.

We could, of course, build the Borromean rings in three dimensions such that they project to such a pattern of circles, while weaving up and down to make the correct over- and under-crossings. A three-dimensional configuration with more symmetry (indeed, with the most symmetry possible) can be built out of three ovals in perpendicular planes. As in Figure 5, the ovals can be rectangles, stadium curves, or ellipses, even arbitrarily close to round circles. But it is impossible to build the Borromean rings—or indeed any Brunnian link—out of round circles in space. (See Ian Agol’s writeup [1] of a slick proof due to Mike Freedman. The paper [13] seems to incorrectly deal only with the case that the three-dimensional configuration has a projection homeomorphic to Figure 2. See also [11].)

2 Tight Knots and Links

Mathematicians usually study links (including knots) topologically, considering as equivalent any two curves which can be deformed into each other, and trying to classify the resulting link types. In contrast, geometric knot theory considers specific geometric shapes for links. For instance, we try to prove that knotted or linked curves have greater complexity, by some geometric measure, than unknots. (An example is the Fáry/Milnor theorem: a knotted curve has at least twice as much total curvature as a round circle.)

Geometric knot theory also looks for optimal shapes for each knot or link. Although there are many

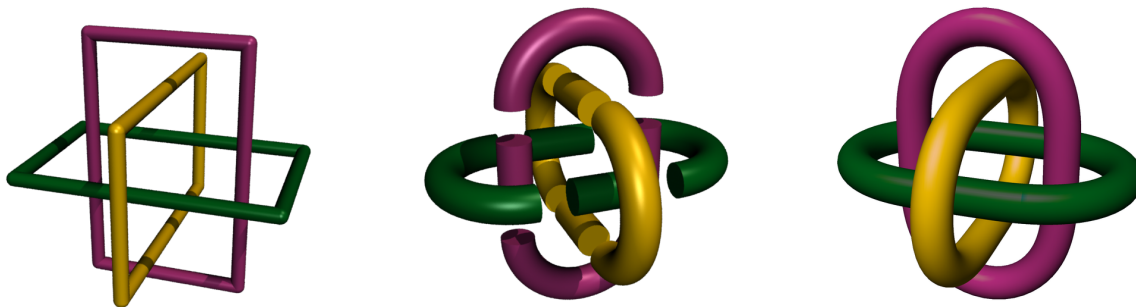


Figure 5: The Borromean rings can be built with pyritohedral symmetry using three ovals in perpendicular planes. Each curve can be a rectangle (left), a stadium curve (semicircles joined by straight segments, center), or a minimizer for Möbius energy (right).

interesting notions of optimality that can be considered here, one which is physically relevant and mathematically challenging is the ropelength problem: we tie the link tight, using the shortest possible length of rope. Mathematically, we minimize the length of a space curve (in the given link type) subject to the condition that a unit-diameter tube around it remains embedded.

It is known [4, 8] that any link has a ropelength-minimizing configuration. Some simple examples of minimizing links [4] are known—in these, each component is convex, planar and built of circles and straight segments. Although minimizers are always $C^{1,1}$ (tangent continuous with Lipschitz tangent vector), these examples show they need not be C^2 (curvature continuous). Despite some good numerical simulations (including [14]) of the tight trefoil, nobody even has a good conjecture of what its exact shape (or that of any tight knot) should be.

Intuitively, in a tight knot, the elastic tension forces trying to reduce length (by moving the curve in its principal normal direction) must be balanced by contact forces, where one part of the tube bumps up against another because two strands of the link have reached distance 1. Our Balance Criterion [3, Thm. 6.1] made this intuition precise and allowed us to compute the exact shape of the tight Borromean rings. (The paper [3] deals with a technically easier variant of the ropelength problem. In work in progress, we extend the balance criterion to the setting of ordinary ropelength. For the Borromean rings, our tight configuration is a critical point for both problems; we expect it is also the minimizer for both.)

The tight configuration [3, §10] has pyritohedral symmetry, as shown in Figure 6: the three components lie in perpendicular planes; there is mirror symmetry across each of these planes and a 3-fold rotation symmetry cyclically permuting the components. The curves are piecewise smooth, but with a total of 42 smooth pieces, some only describable in terms of elliptic integrals. The tight configuration is geometrically very close to (and less than 0.1% shorter than) a configuration where each of the three components is built of unit-circle arcs, two convex and two concave.

3 Pyritohedral Symmetry

Certainly one could make an interesting mathematical video about the theory of tight links, the balance criterion, and examples like the Borromean rings. But since the initial audience for our video about the IMU logo was to be mathematicians from all fields, we decided it would be most appropriate to show only a bit of knot theory and instead draw out the connections to other parts of mathematics.

Visually appealing was an exploration of the pyritohedral symmetry group, named $3\star 2$ in the Conway–Thurston orbifold notation or $m3$ in crystallography. The point groups—the symmetry groups of finite three-

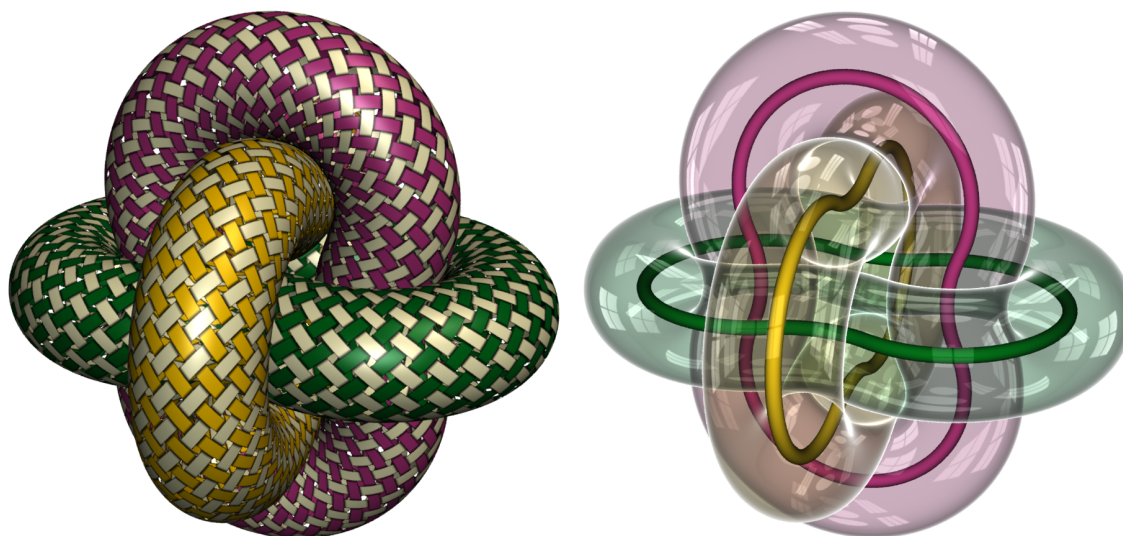


Figure 6: The tight configuration of the Borromean rings, with pyritohedral symmetry, can be rendered in different styles. A woven rope texture (left) suggests the physical process of tying a knot tight. Transparent thick tubes around thin core curves (right) suggest the mathematical model used.

dimensional objects—can be divided into the axial groups (which preserve an axis line in space, and which come in seven infinite families) and the polyhedral groups (of which there are seven). The latter include the full symmetry groups of the tetrahedron, octahedron and icosahedron as well as their rotational subgroups. The seventh polyhedral group, the only one that is neither a rotation group nor a reflection group, is the pyritohedral group, so-named because pyrite crystals often have such symmetry. In cartesian coordinates, the pyritohedral group is generated by sign changes and cyclic permutations of the coordinates (corresponding to mirror reflections across the coordinate planes and 3-fold rotation around the $(1, 1, 1)$ -axis).

To illustrate this pyritohedral symmetry in a video, it is most interesting to have a moving object which keeps the symmetry. We used the unfolding of an octahedron through an icosahedron to a cuboctahedron, popularized by Buckminster Fuller as the “jitterbug” [7] and shown in Figure 7. The audience is led to discover how the Borromean rings arise within the jitterbug motion, as the boundaries of three golden rectangles; this provides a natural transition to the rest of the video.

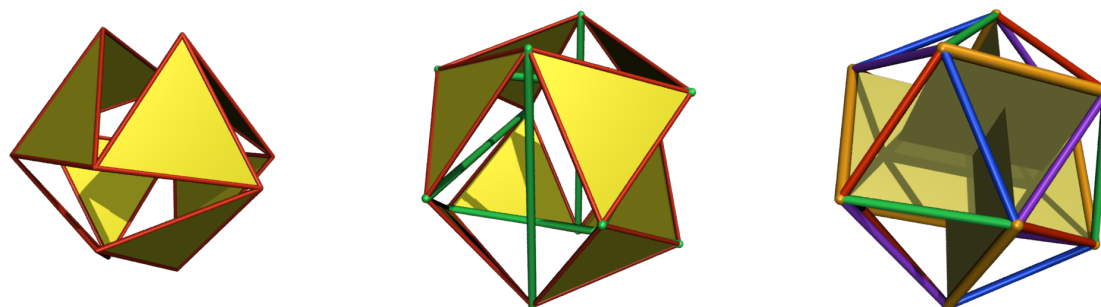


Figure 7: The jitterbug is an unfolding of an octahedron that preserves pyritohedral symmetry (left). If the extra icosahedron edges follow along in the jitterbug, they become the six struts of this tensegrity structure (center), the long sides of three golden rectangles forming the Borromean rings. The pyritohedral group has index five in the icosahedral group, related to a five-coloring of the icosahedron’s edges (right).

4 Making the Video

For an animation like this, telling a mathematical story, previous experience had taught us that the development should be driven mainly by the script. With a rough idea of what kinds of images would be used, we created a story-board including a first draft of the script. After developing an interactive application to animate most of the scenes, we recorded the script and timed it (to a granularity of about one sentence). These timings were used to plan out the graphical animation in more detail. Of course, there was feedback: when certain segments were visually too fast or too slow, we made appropriate edits to the script. But once the final script was recorded, its timings were again used to fine-tune the animation.

Most of the scenes were animated in jReality [12], a 3D scene-graph package for Java written by our group at the Technische Universität Berlin. The compact but flexible scene graphs of jReality can be coupled to a variety of backends, customized for different sorts of rendering. Real-time rendering can be done in JOGL (OpenGL for Java) or in software, while further backends write files—for instance in VRML, PostScript or RenderMan formats—for later processing by other renderers.

Our interactive Java application uses the JOGL backend for real-time playback. We have exported this interactive version of our animation as a Java webstart application, which is easy to download and run on any operating system and is available from the video webpage [10]. Of course, any such OpenGL rendering involves certain quality constraints, such as incorrect transparency calculations due to unsorted polygons, and lack of sophisticated lighting techniques like shadows. Therefore, the final video was produced using jReality’s RenderMan backend: for each frame, a file was output for photorealistic rendering in Pixar’s RenderMan, giving us correct transparency and shadows.

Once we had rendered a still image for each frame of the video (over 5000 in total) these were assembled into a movie using Final Cut Pro. This and the rest of the postproduction was done by Samy Khadem-al-Charieh in the Videostudio at the TU Berlin math department. In the first version of our video, the interlude (zooming in on photos of real-world borromean rings) and the credits were also made in Final Cut Pro.

Built into jReality is support for stereo cameras, so we have also produced a 3D-stereo version of the video, suitable for showing in 3D theaters. Each frame is double-width, with the left-eye view next to the right-eye view. Typically, the computer playing it back will send the two halves of the image to two separate projectors, each aimed through a polarizing filter at the same screen. Audience members see a stereoscopic 3D effect by wearing inexpensive polarized glasses, as in Figure 8.

Producing the stereo version also spurred us to make the credits more visually interesting, as shown in Figure 9: they now scroll along a hyperbolic cylinder that has been placed to fit within the camera frustum. The text, rendered as a flat image, is applied to the cylinder via a texture map. Thus, as the text scrolls



Figure 8: The audience wears polarized glasses for a 3D-stereo presentation of *The Borromean Rings*: *A new logo for the IMU.*



Figure 9: The credits at the end of the video scroll across a hyperbolic cylinder in the 3D scene.

upwards, it first moves in towards the viewer from far away, becomes closest as it crosses the middle of the screen, and then regresses from view at the top.

The motions in the video were controlled by a simple set of Java classes for keyframe animation. Between the keyframes, the value of each animated parameter gets interpolated linearly or cubically. The parameters can be real numbers (controlling, for instance, tube thickness, or progress within the jitterbug motion), colors (four-channel, including transparency), or Euclidean motions (of the camera or objects in the scene). Many of our parameters remained constant for some time, then changed to a new constant value; in such cases, we typically used the Hermite form of cubic interpolation, with zero derivative at the beginning and end of the transition. This gives a C^1 function of time, making the transitions visually gentle.

Since most of the animation was driven by the need to show precise mathematical ideas, we set the keyframe values by hand in the source code, rather than using a graphical user interface. This also allowed us to very easily make the final timing adjustments to synchronize the animation to the recorded script.

The video consists of two main animated sections, separated by an interlude showing photos of the borromean rings as used outside mathematics. The first section illustrates pyritohedral symmetry, largely through the jitterbug. Cinematically, this section is quite tame, with a fixed camera position. Here we made use of the 24-element pyritohedral symmetry group to build a scene graph that applies 24 different Euclidean transformations to a small collection of geometric primitives. For the jitterbug, for instance, the fundamental domain contains one-third of one face of the initial octahedron: in our pictures, a yellow 120-30-30 triangle with a red tube along its long edge. The pyritohedral symmetry transformations were implemented within a Java package for discrete groups developed by one of us (Gunn); compare [9].

When a single object is included several times into a scene, with different transformations, the scene-graph structure of `jReality` allows different appearances to be applied to the different copies. Consider, for instance, the scene showing how the pyritohedral group sits inside the icosahedral group, using the five-coloring of the icosahedral edges. We start with three perpendicular golden rectangles, whose ends are the six edges of a single color, and which are rendered as above as 24 copies of a single fundamental unit. This whole configuration is then included in the scene graph as many as five times; the most recent copy rotates into place while its color and tube radius vary in time. (In `jReality`, an edge is automatically drawn as a tube; the tube radius is part of the edge's appearance, and thus can be specified separately for different copies. If the scene graph instead explicitly included the cylindrical tube, this would not be possible.)

The video has a fixed shaded background, seen in Figure 3, which comes simply from a rectangle placed behind the rest of the scene; the graphics system interpolates bilinearly between the four colors specified at its corners. Most of the other figures in this paper are also stills from our video, but we have removed the shaded background for the benefit of those reading a black & white printed copy.

The last section of the video explores different possible shapes for the Borromean rings, ending with the tight form used in the IMU logo. While most of the shapes do have pyritohedral symmetry, we typically implemented them making use only of the three-fold rotation symmetry that cycles the three rings. (In most cases we had a polygonal approximation to one whole ring, and it was easier to read that in as a whole, rather than to extract one fundamental domain for its four-fold mirror symmetry. Also, the use of the rope textures described below breaks the mirror symmetry.) We again used the feature of `jReality` mentioned above—allowing a different appearance for each transformed copy of an object—to give the three rings distinct colors (and later, different textures or transparency).

One scene (Figure 5, right) shows a configuration that minimizes a knot energy based on spreading electric charge along the knot. This scene, adapted from the video [15], was rendered in `avn`, completely independent of `jReality` and `RenderMan`. We adapted `avn` to use the color scheme and background we had chosen for our video (as well as to optionally produce stereo output), but the attentive viewer will notice the slightly different lighting and shading model in this scene. Originally developed for [16], `avn` is an interactive viewer which includes its own keyframing mechanisms [6]. Here, these were used to smooth the camera path as well as the path through the group of Möbius transformations, which illustrates the Möbius

invariance of this knot energy. (The IMU logo itself, Figure 1, was also rendered in `avn`, which can do its own high-resolution rendering or can export to `RenderMan`.)

In the final scenes, showing the tight Borromean rings, our main technical challenge was how to depict a tight link. We always render a space curve by drawing a tube of constant width around it. When the curve represents a tight knot, it is of mathematical interest to show the thick tube, the one that bumps up against itself. But the large amount of self-contact, of course, tends to block the view. We found it helpful to show several different renderings of the same scene, as in Figure 6. Sometimes we draw a thinner tube, as in the IMU logo itself; sometimes we apply a woven-rope texture (described below) to the thick tube; and sometimes we make the thick tube transparent, with a thin opaque tube in the middle to show the core curve.

As we mentioned above, although the tight configuration of the Borromean rings has a complicated geometry, it is extremely close to a configuration made of a small number of circular arcs. While the IMU logo (Figure 1) is rendered from a fine polygonal approximation to the true tight curve, for the video, we “cheated” by rendering tubes around the nearby circular arcs, which are thus parts of round tori.

Indeed, to obtain efficient rendering of the highest possible quality, we took advantage of the fact that `RenderMan` has built-in primitives for torus surfaces. The `jReality` `RenderMan` backend allows the user to override the default output for part of the scene graph, and instead output an arbitrary `RenderMan` command, specified within the appearance attributes for that node. Here, we simply output a torus primitive instead of the polyhedral approximation used in `jReality`. Similarly, for the depiction of stadium curves, in `RenderMan` we can draw the tube around each curve as two half tori and two cylinders. Indeed, we implemented the momentary shortening of the cylinders (as in Figure 5, center) only for the `RenderMan` backend, not in the interactive application.

For the scene with transparent thick tubes around opaque thin tubes (Figure 6, right), we used the soap-film shader for `RenderMan` originally described in [2]. This shader implements Fresnel’s laws for thin-film optics, thus creating much more realistic transparency than simple alpha-channel schemes do. The important qualitative feature is that each patch of surface reflects or transmits all the light it receives; when the surface is seen obliquely, most light is reflected, while when it is seen head-on, most is transmitted.

Other scenes use a procedurally generated texture that simulates the appearance of woven rope, as in Figure 10. This texture incorporates transparency, letting us see between the woven strands. The procedure that generates the woven-rope texture has several parameters, allowing the user to control the colors of the two woven bands, the amount of implied shadow rendered where the bands cross, the (usually transparent) color of the gap between the bands, and the relative width of the bands and gaps. For appropriate choices of the parameters, the texture is a constant color; starting there, we can gradually fade to the rope texture we want, including the transparent gaps. Later, by increasing the gap width, we create the effect of “dissolving” two components of the Borromean rings, as in Figure 10, giving us good view of the shape of the remaining

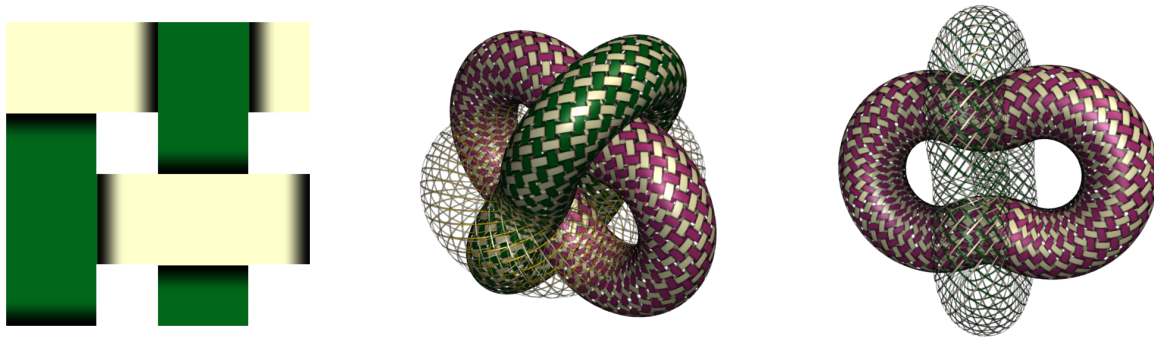


Figure 10: One of the parameters for the woven-rope texture (left) controls the gap width, allowing us to successively “dissolve” the woven strands on two components of the Borromean rings.

component. These scenes use a free-flowing camera motion, splined between keyframes chosen to emphasize certain symmetric views.

At the end of the video, we fade to the IMU logo itself. The logo was rendered using a distant camera with a small field of view, giving (almost) an orthogonal projection: this yields a more symmetric two-dimensional picture, where for instance the constant-width tubes appear as constant-width bands in the image plane. For most of our video, we instead adopted a 30° field-of-view (measuring the full vertical angle), which is similar to that given by a “normal” 50 mm lens on a standard 35 mm camera. The human visual system is not usually bothered by viewing pictures with the wrong field-of-view; indeed feature films commonly cut between wide-angle and telephoto shots, and they work well enough both for those who choose to sit in the front row and for those who sit way in the back of the theater. But to create a good three-dimensional effect from stereoscopic pictures, it becomes more important to render with the correct field-of-view, and 30° was our guess of the apparent size of the screen for a typical audience member in a typical auditorium. To fade from this to the logo, we did a Hermite interpolation of the field-of-view angle, while moving the camera away at a speed calculated to (approximately) maintain the rendered size of the Borromean rings.

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