# Polyhedra with Equilateral Heptagons

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Open Python Application Readme

### Abstract

Seven has a religious and cultural meaning in our society. For instance, seven is the number of days in a week, and in the art of music seven is the number of notes on a musical scale. The fact that seven is so interwoven in our society serves as an artistic source of inspiration for the polyhedra presented in this paper: they all contain equilateral heptagons. These are derived by truncating kites in polyhedra that appear in a method of morphing between pairs of dual regular polyhedra. Also presented are some paper models of these polyhedra.

## Introduction

It seems that heptagons do not fit very well in 3D space. Except for the heptagonal prisms and anti-prisms the uniform polyhedra [1] do not have regular heptagons. Neither do any of the Johnson solids [3]. Mason Green, on the other hand, presents some remarkable polyhedra [2] that have cyclic symmetry and consist of regular faces including heptagons. The polyhedra presented in this paper all contain *equilateral* heptagons. They share more properties and they can all be summarised as a set of requirements:

**Requirement 1** The polyhedron contains at least one heptagon.

**Requirement 2** The polyhedron belongs to one of the following symmetry groups: $A_4$, $A_4 \times I$, $S_4A_4$, $S_4$, $S_4 \times I$, $A_5$, or $A_5 \times I$. [1] The reason for leaving out the cyclic and dihedral symmetry groups is a matter of taste: they can be interpreted as flat symmetries extended to the the third dimension, while the previous group of symmetries can be interpreted as pure 3D symmetries.

**Requirement 3** The polyhedron is equilateral, which means all faces are equilateral. This requirement has an artistic background. It unifies all faces in the resulting polyhedron so that the faces that are not heptagons are related to the heptagons by their edge length.

**Requirement 4** The heptagon(s) are convex. This requirement was added to limit the set.

## Deriving Polyhedra

One way to search for polyhedra in this set is as follows. Position one or more regular heptagons in 3D space and use the symmetry operations of a desired symmetry group to obtain a set of heptagons belonging to exactly that symmetry group. The gaps that are left can be filled with triangles. The resulting polyhedron would not fulfil requirement 3; however, the position of the initial heptagon(s) can be varied, and in such a way special positions might be found for which all faces become equilateral. The beauty of this method is that all heptagons will be regular; the disadvantage is, however, that the chance of finding equilateral polyhedra seems slim.

[1] $A_4$, $S_4$, and $A_5$ only contain the rotational symmetries of the tetrahedron, the cube, and the icosahedron respectively. For the symmetry groups $A_4 \times I$, $S_4 \times I$, and $A_5 \times I$ the central inversion is included as well. $S_4A_4$ is the complete symmetry of a tetrahedron.

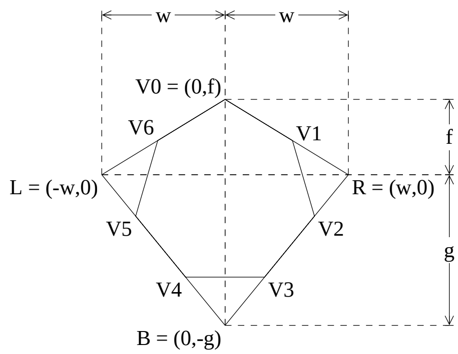


**Figure 1**: Equilateral Heptagon Constructed from a Kite

A more successful way of finding polyhedra belonging to this set is to truncate the faces of polyhedra that consist of kites. Kites can be truncated so that equilateral heptagons are obtained. This is shown in Figure 1. Not all kites can be truncated this way. If distance $g$ increases, $V_1$ will move towards $R$ until $V_1$ ends up exactly at $R$, and the heptagon becomes a pentagon. Continuing after this point will result in a concave heptagon. On the other hand, decreasing $g$ will push $V_2$ towards $R$ until $V_2$ ends up exactly at $R$, and the heptagon becomes a pentagon. Again, continuing after this point will result in a concave heptagon.

A mathematical expression can be derived that can be used to calculate the vertices of the heptagon. Given the orientation of the kite as illustrated in Figure 1 the following holds:

$$\begin{aligned} V_0R: \quad y &= -\tfrac{f}{w}x + f \\ BR: \quad y &= \ \ \tfrac{g}{w}x - g \end{aligned}$$

If $V_i = [x_i, y_i]$ for $i \in [0,6]$, then

$$\begin{aligned} x_1 &= \frac{f+g}{qk+r+\sqrt{r^2+2k-k^2}} \\ x_2 &= kx_1 \\ x_3 &= mx_1 \end{aligned}$$

with

$$r = \frac{f}{w} \qquad q = \frac{g}{w} \qquad m = \frac{1}{2}\sqrt{1+r^2} \qquad n = \frac{1}{2}\sqrt{1+q^2} \qquad k = (1+\frac{1}{n})m$$

The other vertices $V_4 - V_6$ are implicitly defined by the bilateral symmetry. For some kites the truncation can be done in two ways: either by truncating $B$, $L$, and $R$ as shown in Figure 1, or by truncating $V_0$, $L$, and $R$, which results in a different orientation of the heptagon; a heptagon that is upside down compared to the one in Figure 1.

A known technique for morphing a regular polyhedron into its dual is by tilting quadrilaterals [5]. This morphing results in an infinite set of polyhedra consisting of kites. The kites can be truncated to equilateral heptagons, which leaves gaps in the polyhedron. The gaps that are obtained by truncating corner $B$ (or $V_0$) of the kite can be filled with a regular polygon; the gaps that are obtained by truncation of $L$ and $R$ consist of four vertices and can be filled with a pair of triangles in two different ways. These triangles are, in general, isosceles, but for special morphing positions they can become equilateral. Another possibility is that they become coplanar and form a rhombus. As a result there are three possible special positions per dual pair

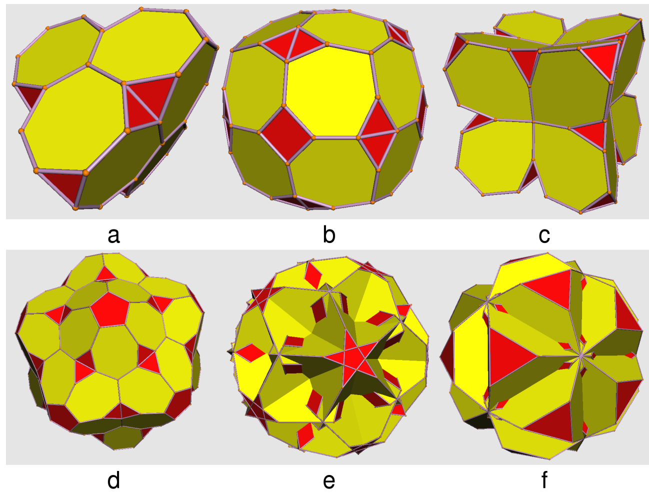


**Figure 2**: Examples of Equilateral Polyhedra with Heptagons

and per orientation of the heptagon. Note that for the tetrahedron there is only one orientation, since it is self-dual.

To investigate the dynamics of morphing and truncating I wrote a software program that generates VRML animations. It uses numerical methods to calculate the special positions for which all faces become equilateral. For these positions the program generates 3D modelling files in VRML or OFF format and Postscript templates for building a model of the polyhedron.

Figure 2 shows some examples of special morphing positions for which all faces are equilateral after truncation. The pictures were created with help of the OFF file viewer Antiview in the software package Antiprism [4]. The polyhedron in 2a is derived from a tetrahedron, and the polyhedra in 2b and 2c are both derived from morphing an octahedron into a cube. However, different truncations of the kite would have to be chosen. In 2b the vertices on a 4-fold axis are truncated, resulting in squares, while in 2c the vertices on a 3-fold axis are truncated, resulting in the equilateral triangles sharing a 3-fold axis. For 2b the morphing is somewhere between the cube and octahedron, while in 2c the morphing continued beyond the cube. The polyhedron in 2d is the result of morphing and truncating an icosahedron-dodecahedron pair, and 2e and 2f are derived from morphing a dual pair consisting of a great icosahedron and a great stellated dodecahedron. However, in 2e the vertices on a 5-fold axis are truncated, while in 2f the vertices on a 3-fold axis are truncated. This results in regular pentagrams in 2e and in the equilateral triangles sharing a 3-fold axis in 2f. In 2b it seems that pairs of triangles are coplanar, however, for the position for which the triangles are actually coplanar that diagonal of the rhombus has a length of approximately $1.02u$, where $u$ equals to the edge length. For 2e it is worth mentioning that a pair of triangles almost form a pair of faces of a regular tetrahedron; the distance between the unconnected vertices is approximately $1.07u$.

Figure 3 shows examples of models of other special positions that were found. Model 3a is derived from an icosahedron, and pairs of triangles form rhombi that almost look like squares. Model 3d also contains rhombi and is derived from a cube, while model 3b is derived from a tetrahedron. The two other models are both derived from a great dodecahedron (or small stellated dodecahedron). For the models in Figure 2d and 3a the same kite vertices were truncated, though for the latter rhombi are obtained. The same holds true for the models in Figure 2b and 3d. Similarly, for the models in Figure 2a and 3b, and the models in Figure 3c and 3e the same kite vertices were truncated, though different pairs of triangles were chosen.

All models but one were built out of cardboard faces. Model 3c also uses a transparent plastic to be able to show the pentagons that completely end up inside the polyhedron. For this model it is interesting to mention that the heptagons almost look like regular heptagons. The rounded values for the angles at the

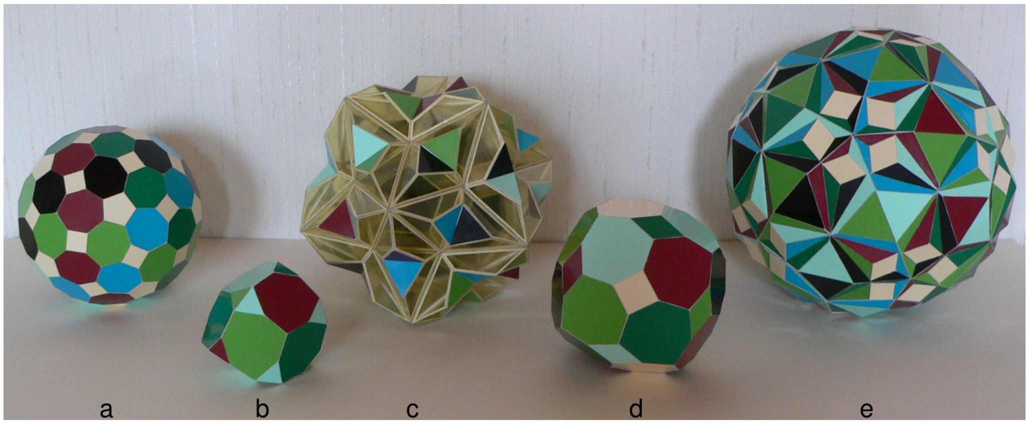


**Figure 3**: Five Models of Equilateral Polyhedra with Heptagons

corners are 130, 133, 126 ,126 ,133 ,130 and 123 degrees. The choice of colour is an artistic element of the creative work. To express that all models are related the colours of the cardboard faces were taken from one set of seven colours. It holds true for every model that the colouring itself is related to seven as well, either by using the full set of seven colours in a balanced way with regard to their symmetry, or by using a subset in such a way that seven faces have the same colour and any set of faces with the same colour can be mapped onto any other set of faces with the same colour by an operation of the symmetry of the polyhedron. Models 3a, 3c and 3e are examples of the former, models 3b and 3d are examples of the latter.

## Conclusions and Further Work

The paper presented some interesting examples of equilateral polyhedra that contain heptagons. These were derived by truncating kites of morphed regular polyhedra. For a subset of the polyhedra that the method generates holds that except for the heptagons, all faces are regular. Some cardboard models were also presented; one of which used a transparent material to show the faces that completely end up inside. A possible application of the presented models can be found in the area of decorations of e.g. lamp shades. The method generates more polyhedra that fulfil the set of requirements: e.g. other interesting polyhedra that would require transparent material because edges or faces end up completely inside the polyhedron. One of these requires seven layers.

Other methods for finding polyhedra belonging to the set can still be explored. For instance the method described in the beginning of the paper where one or more regular heptagons are positioned in 3D space and the symmetry operations of the desired symmetry group are used to generate a polyhedron.

## References


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