

# Can Geometry Create Art?

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## Abstract

The work of M.C. Escher shows that the best artists can raise to an admirable degree of perfection some simple geometric facts. This talk, makes a journey into planar and three dimensional theory and shows geometric optimization problems where the theory alone provides the aesthetically pleasing solutions.

## 1 Introduction

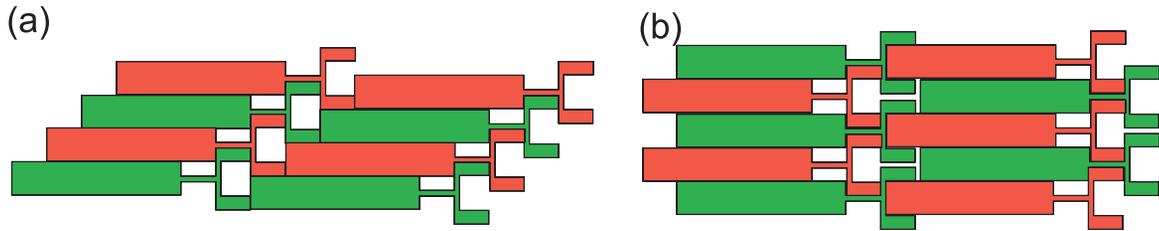
The author believes that results of geometric optimization problems can be inspirational for artists. In other words, typical research work in the area of discrete geometry can provide motives, which otherwise are hard to come up with. Most of the them stimulate thinking. These constructions complement the common belief of most mathematicians, which says that good mathematics, i.e. the creation of elegant arguments, is a form of art. The talk starts with an overview of discrete geometry, which is followed by an array of geometric questions, to illustrate how geometry provides unique arrangements.

## 2 Overview of the research in discrete geometry

*Intuitive (elementary) geometry* is one of the oldest branches of mathematics. Later at the beginning of the 19th century geometry was commonly viewed as a teaching tool to introduce mathematical thinking. In the fifties it started to flourish again. Mainly the works of L. Fejes Tóth [6] and C. A. Rogers [10] brought attention to those questions which were already considered by Newton, Gauss, Minkowski and Hilbert. Initially they used the term *discrete geometry* to refer to the theory of packings and coverings. At the same time another branch, called *combinatorial geometry*, gained lot of interest too. Problems raised by P. Erdős shaped the research in that area. The relatively recent development of *computational geometry*, a field created by the needs of computer science, gave discrete and computational geometry new significance and boosted its further development. The properties of optimal discrete arrangements are extremely important in computational geometry. Results of this type gave theoretical foundation for finding efficient computational algorithms. Most of the problems studied in discrete geometry concern the properties of arrangements of basic geometric figures like points, lines, circles convex sets etc.

The basic problems in the theory of packing and covering are the determination of the most efficient packings (arrangements without overlap) and coverings (arrangements whose union is the entire space) of congruent replicas of a given set  $K$  in the space. The usual measure of the efficiency of an arrangement is the *density*. The monographs by L. Fejes Tóth [6], and C.A. Rogers [10] contain a description of the classical results. A detailed description of the current state of the theory of packing and covering can be found in the problem collection book of J. Pach, P. Brass and L. Moser [5]. We do know, due to Rogers, that the maximum density among all packings in the plane with translates of a given convex set is attained in a lattice arrangements of translates.

**Example 1.** *A construction of the author shows that this does not remain so if the condition of convexity is omitted (Figure 1).*



A polygon whose translates can be packed more densely (b), than the polygon's densest lattice packing (a).

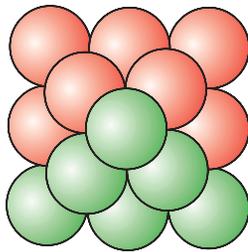
Figure 1

In three (and higher) dimensions, the packing and covering problems are much more difficult than their planar versions, even in the special case in which only translates of a convex body are used, and even if only lattice-type arrangements are considered.

### 3 Optimal arrangements in 3-space

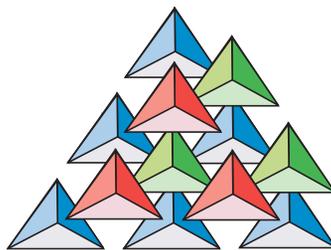
From all problems concerning packing and covering it was the sphere packing problem (also called Kepler's conjecture), that is the determination of the packing density of the sphere, which recently attracted the most attention (for the computer aided proof see T. Hales [8]) (Figure 2). The following problem is also natural:

**Example 2.** Find reasonably good upper and lower bounds for the density of the densest packing of translates in case of a specific class of convex bodies. This class can be the family of central symmetrical convex bodies, the family of cylinders, the family of cones (Figures 3-7).



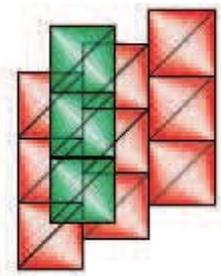
The densest sphere packing.

Figure 2



The densest lattice packing of tetrahedra.

Figure 3



The densest lattice packing of double clones.

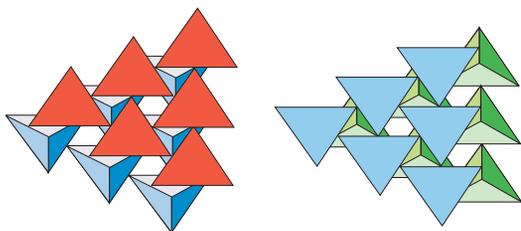
Figure 4

A joint study of the author and W. Kuperberg [2] produced a number of results concerning packings of compact convex cylinders in  $E^3$  (i.e., products of a convex plane body with an interval).

**Example 3.** Somewhat surprisingly, it turned out that, for certain cylinders, stacking them in infinite "beams" is not always the most economical way of packing (Figures 8-9).

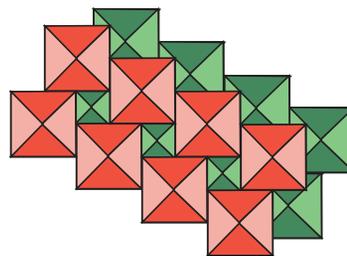
Concerning ellipsoids, another surprise was discovered [3]):

**Example 4.** Certain congruent ellipsoids in  $E^d$  (for  $d \geq 3$ ) can be packed in a nonlattice-like manner more densely than in a lattice-like manner.



Two layers of congruent tetrahedra, which can be stacked to form a packing with density  $2/3$ .

Figure 5



A square pyramid packing, conjectured to be the densest among all translational pyramid packings.

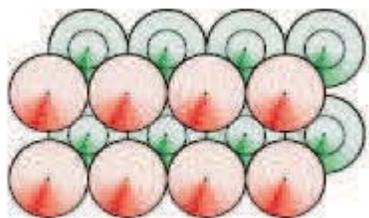
Figure 6

From Hales' proof of Kepler's conjecture it follows that the ellipsoid packing we found is denser than *any* sphere packing, not just lattice-like. Thus, the space-packing problem for ellipsoids is strikingly different from the analogous plane-packing problem for ellipses(see [6]).

Some of the previous results in plane geometry have not been generalized to higher dimensions.

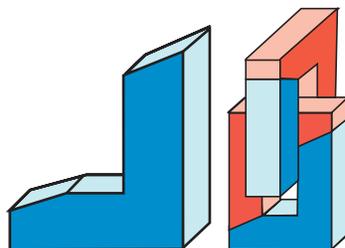
**Example 5.** *The restriction to arrangements of translates of a set seems to be a significant simplification (Figures 5-7).*

Still, we do not know about any convex body besides tiles, the cylinder, the sphere and the truncated rhombic dodecahedron whose densest packing with translates is known, i.e. whose *translational packing density* is known. In particular, this maximum density is not known in the case of the tetrahedron.



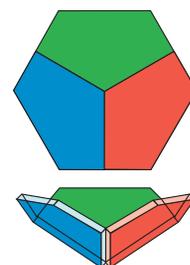
The conjectured densest translational circular cone packing.

Figure 7



A compact nonconvex cylindrical tile, whose base is not a tile.

Figure 8



A convex nontiling pentagon over which short cylinders almost tile the space.

Figure 9

#### 4 Littlewood's problem

The following problem was posed by Littlewood [9]. What is the maximum number of congruent infinite cylinders that can be arranged in 3-space so that any two of them are touching? Is it 7?

The analogous problem for concerning circular cylinders of finite length became known as a mathematical puzzle due to a popular book of Martin Gardner [7]: Find an arrangements of 7 cigarettes so that any two of them touch each other. The question whether 7 is the largest such number is open. The original question of Littlewood is indeed very interesting and surprisingly hard. A. Bezdek [4] proved that this number can not exceed 24. There are several possible types of arrangements of six mutually touching infinite cylinders (see the survey of Brass, Pach and Moser [5], and the ones we know are flexible with one degree of freedom, hence suggesting a possibility for extending the configuration with a seventh cylinder.

In the early 1990's, W. Kuperberg assembled a nice, symmetrical arrangement of eight pencils (see Figure bellow), showed the physical model to people (i.e. he did not specify all the parameters), and asked them to

decide whether the pencils are mutually touching or not. It was such a close call, that people wondered if for specific choices of parameters the pencils can indeed be mutually touching. In a joint paper with G. Ambrus [1] we recently showed that this is not the case. Interestingly, one can require mutual contact for any four of the 8 cylinders, but one can not do the same for certain five cylinder subfamilies.



Figure 10

**Example 6.** *Among the eight pencils (Figure 10) of the configuration of W. Kuperberg, there are two not sharing a common point.*

Theorem 1 is unusual in the sense that it is a precise mathematical statement about a model which itself is not described mathematically. How will we handle this obstacle rigorously? First we stated couple of properties which are undoubtedly satisfied by the 8 pencil model, and based on them, we proved some precise lemmas and finally proved the needed theorem.

## References

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