Metamorphosis in Escher's Art

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Abstract

M.C. Escher returned often to the themes of metamorphosis and deformation in his art, using a small set of pictorial devices to express this theme. I classify Escher's various approaches to metamorphosis, and relate them to the works in which they appear. I also discuss the mathematical challenges that arise in attempting to formalize one of these devices so that it can be applied reliably.

1. Introduction

Many of Escher's prints feature divisions of the plane that change or evolve in some way [11, Page 254]. The most well-known is probably *Metamorphosis II*, a long narrow print containing a variety of ingenious transitions between patterns, tilings, and realistic scenery. Escher was quite explicit about the temporal aspect of these long prints. He would not simply describe the structure of *Metamorphosis II* – he would narrate it like a story [5, Page 48].

I am interested in the problem of creating new designs in the style of Escher's metamorphoses. More precisely, I would like to develop algorithms that automate aspects of the creation process. To that end, I have studied the devices Escher used to carry out his transitions, with the ultimate aim of formalizing these devices mathematically. In this paper, I present my taxonomy of transition devices (Section 2) and provide a cross-reference to the Escher works in which they appear. I then discuss what is known about the transition types (Section 3) and focus on one type in particular—Interpolation—which shows the most promise for a mathematical treatment.

2. Escher's transitions

A survey of Escher's work (as collected by Bool et al. [1]) turns up 18 works employing some kind of transition device. By studying these works, I have identified six categories of transition. *Metamorphosis II* serves as a kind of atlas, as it incorporates all six varieties. They are as follows:

- **T1**. **Realization:** A geometric pattern is elaborated into a landscape or other concrete scene. In *Metamorphosis II*, a cube-like arrangement of rhombi evolves into a depiction of the Italian town of Atrani.
- **T2**. **Crossfade:** Two designs with compatible symmetries are overlaid, with one fading into the other. Escher applies this device sparingly, using it in *Metamorphosis II* and *III* to transition from a rectilinear arrangement of copies of the word "metamorphose" into a checkerboard (and later, to make the reverse transition).

T3. **Abutment:** Two distinct tilings are abruptly spliced together along a shared curve. The transition works when the two tilings have vaguely similar geometry and can be made to abut one another without too much distortion. Escher uses this device exactly once, to transition from hexagonal reptiles to square reptiles in *Metamorphosis II*. (Later, he embedded the same sequence into the larger *Metamorphosis III*.)

T4. **Growth:** Motifs gradually grow to fill the negative space in a field of pre-existing motifs, resulting in a multihedral tiling. The new motifs need not occupy all the empty space; in *Metamorphosis II*, red birds grow to occupy half the space between black birds. When the two sets of motifs finally fit together, they leave behind a white area in the form of a third bird motif.

T5. **Sky-and-Water:** This sort of transition starts with copies of some realistic shape A, ends in copies of another realistic shape B, and moves between them by passing through a tiling from two shapes that resemble A and B. in Escher's *Sky and Water*, realistic birds above encounter realistic fish below, using the tiling first recorded as Number 22 in his notebooks [11] as an interface.

T6. **Interpolation:** A tiling evolves into another tiling by smoothly deforming the shapes of tiles. Escher used this device to change simple tilings into his familiar interlocking animal forms (for example, squares into reptiles in *Metamorphosis II*, and triangles into a variety of forms in *Verbum*). In some cases (such as the print *Liberation*) the animal forms are then permitted to escape from the tiling.

Given this vocabulary of pictorial devices, the sequence of transitions in *Metamorphosis II* might then be read from left to right as

- **T2** (copies of "metamorphose" into a checkerboard)
- **T6** (a checkerboard into a square arrangement of reptiles)
- **T3** (square reptiles into hexagonal reptiles)
- **T6** (hexagonal reptiles into hexagons)
- **T1** (hexagons into a honeycomb with bees)
- **T5** (bees into fish)
- **T5** (fish into black birds)
- **T4** (black birds into birds of three different colours)
- **T6** (birds into a cube-like arrangement of rhombi)
- **T1** (rhombi into the town of Atrani, which then becomes a chessboard)
- **T1** (a chessboard into an orthographic checkerboard — a Realization in reverse)
- **T2** (a checkerboard into copies of "metamorphose")

Likewise, *Metamorphosis III* contains over 20 transitions according to the classification presented here.

Inspired by the cross-reference provided by Schattschneider for Escher's periodic drawings [11], Table 1 presents a concordance between the six transition devices and the works in which they appear.

The table does not record the particular manner in which the transitions are carried out. Given Escher's penchant for interpreting these designs as stories, most transitions are arranged in a linear progression

Table 1 : *A concordance between Escher's transition devices and the works in which they appear. The catalogue numbers in parentheses refer to the compilation of Escher's works by Bool et al. [1].*

		Transition type					
Title	cat.	**T1**	**T2**	**T3**	**T4**	**T5**	**T6**
Metamorphosis I	298	•					•
Development I	300						•
Day and Night	303					•	•
Cycle	305	•					•
Sky and Water I	306					•	
Sky and Water II	308					•	
Development II	310						•
Development II (first version)[a]	310a						•
Metamorphosis II	320	•	•	•	•	•	•
Verbum	326					•	•
New Year's 1949	360					•	
Horses and Birds	363					•	
Fish and Frogs	364					•	
Butterflies[b]	369						•
Liberation	400						•
Regular Division I	416						•
Fish and Scales	433						•
Metamorphosis III	446	•	•	•	•	•	•
Painted Column[c]						•	

[a]Escher carved this woodblock but never printed it.
[b]See also Escher's watercolour painting of Butterflies [10] (no catalogue number).
[c]No catalogue number; a painted concrete column in Haarlem [11, Page 260].

horizontally or vertically, though occasionally they operate radially. For example, *Verbum* is built upon a single tiling by equilateral triangles. Interpolation happens outward from the centre to six realistic animal forms on the edges of a hexagon; six instances of Sky-and-Water occur around the hexagon's circumference.

Furthermore, some choices in the table are open to debate. Should the building in the top half of *Cycle* be taken as a Realization of the tiling below, or do the two merely Abut? Surely not every juxtaposition of a tiling with realistic imagery should be considered a Realization; although three dimensional forms emerge from a printed page in *Reptiles* (cat. 327), the tiling itself does not undergo any kind of transformation. As another example, note that there is some overlap between Interpolation and Sky-and-Water. The special case of tiles evolving into realistic forms and escaping the tiling is very similar to half of a Sky-and-Water transition. This case might indeed be best separated out into a seventh transition type (for which I would suggest the name Liberation, after the print of the same name).

3. The mathematics of deformation

Much scholarly work has sought either to analyze the mathematical structure of Escher's work [2, 4, 11] or to synthesize new designs inspired by it [3, 13]. We might then wonder to what extent the transition types of the previous section might serve as a basis for creating new geometric metamorphoses. As a researcher in computer graphics, I envision a "metamorphosis toolkit", a suite of algorithms that puts these transitions under the control of a designer.

Clearly, there are many challenges to be met in formalizing each of the six transition types. The prevalence of Sky-and-Water and Interpolation suggests that these two should be tackled first. In earlier work, I showed how an optimization technique for discovering Escher-like tessellations automatically [8] could be extended to produce Sky-and-Water designs [9]. In this section I turn to Interpolation transitions, which I formulate as a problem in tiling theory.

Given two tilings $\mathcal{T}_1$ and $\mathcal{T}_2$, Interpolation asks for a smooth geometric transition between the two tilings. Presumably, a one-to-one correspondence is established between the tiles of $\mathcal{T}_1$ and $\mathcal{T}_2$, and as a parameter t moves from 0 to 1, each individual tile gradually deforms from its $\mathcal{T}_1$ shape to its $\mathcal{T}_2$ shape. Like Escher, we seek to carry out this deformation across a region of the plane. We can also consider a temporal variation, in which we construct a continuous animation from $\mathcal{T}_1$ to $\mathcal{T}_2$.

In addition to Escher's art, we can also turn to William Huff's parquet deformations as a source of inspiration. Huff, a professor of architectural design, invented parquet deformations and assigned the drawing of them to his students. They were later popularized by Hofstadter in *Scientific American* [7, Chapter 10]. Huff was inspired directly by Escher's Metamorphoses. He distilled the style down to an abstract core, considering only interpolation transitions, and favouring abstract geometry rendered as simple line art rather than Escher's decorated animal forms. As reported by Hofstadter, Huff further decided to focus on the case where $\mathcal{T}_1$ and $\mathcal{T}_2$ are "directly monohedral," in the sense that every tile is congruent to every other through translation and rotation only. We may also assume he had only periodic tilings in mind. Finally, he asked that in the intermediate stages of the deformation the tile shapes created could each be the prototile of a monohedral tiling. Hofstadter amends this last rule, pointing out that some deformation might be necessary to make the intermediate shapes tile; this amendment is in need of a mathematically rigourous treatment.

Inspired by Escher's metamorphoses and by parquet deformations, I formalize the problem of Interpolation in terms of the theory of isohedral tilings [6, Chapter 6]. Isohedral tilings correspond well to an intuitive notion of regularity in tessellation. They are expressive enough to express a wide range of shapes including Escher's periodic drawings, and admit a compact symbolic description that makes them ideal for implementation in software. Therefore, we express the Interpolation problem as follows: given isohedral tilings $\mathcal{T}_1$ and $\mathcal{T}_2$, called the "key tilings", construct a smooth spatial or temporal deformation between them.

Aside from the relative difficulty of temporal versus spatial transitions, there is a succession of increasingly complex cases to consider, which depend on the relationship between $\mathcal{T}_1$ and $\mathcal{T}_2$:

Case 1. The key tilings are of the same isohedral type and have congruent arrangements of tiling vertices (points where three or more tiles meet).

Case 2. The key tilings are of different isohedral types and have congruent arrangements of tiling vertices.

Case 3. The key tilings are of the same isohedral type.

Case 4. The key tilings are of the same topological type.

Case 5. The key tilings are arbitrary isohedral tilings.

The first case is easily solved temporally or spatially. When the tiling vertices are congruent, there is a rigid motion that maps the tiling vertices of $\mathcal{T}_2$ onto those of $\mathcal{T}_1$. The registration afforded by this rigid motion reduces the general interpolation of tilings to interpolation of curves joining tiling vertices. Any algorithm that interpolates continuously between two paths can be applied to effect a smooth transition. Two

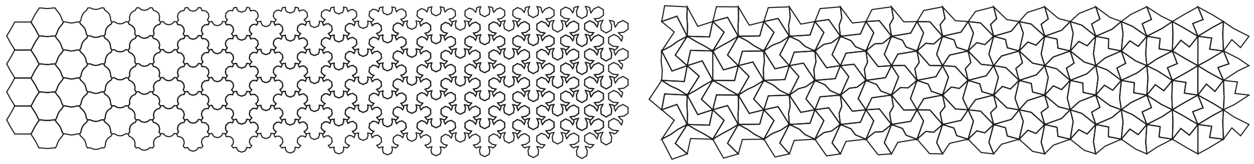

Figure 1 : *Examples of parquet deformations from Case 1, in which the key tilings have the same isohedral type and congruent tiling vertices. The left and right designs are based on isohedral types* IH18 *and* IH88*, respectively.*

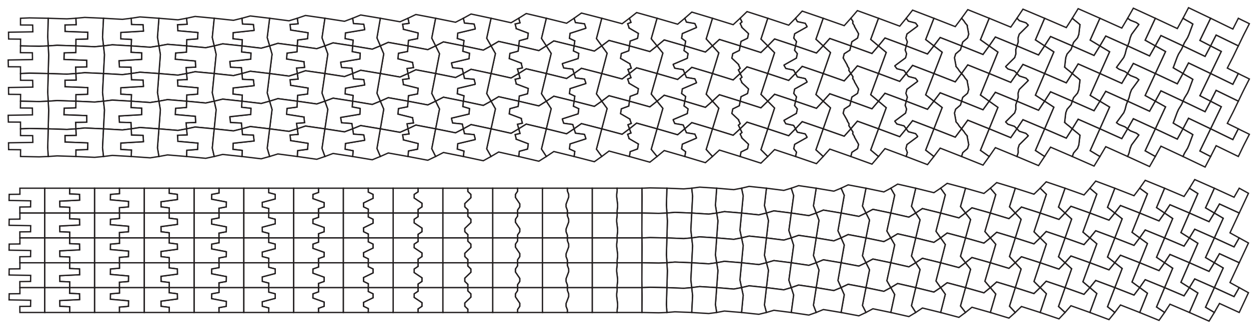

Figure 2 : *Examples of parquet deformations from Case 2, in which isohedral types differ but tiling vertices are congruent. The tilings are of type* IH50 *on the left and* IH61 *on the right. The top row blends corresponding edges directly, leading to two incongruent families of intermediate shapes. The bottom row avoids this problem by passing through the underlying Laves tiling.*

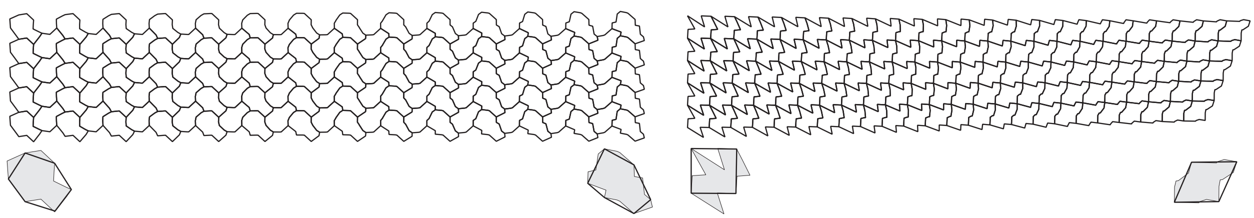

Figure 3 : *Examples of parquet deformations from Case 3, in which both key tilings have the same isohedral type but tiling vertices are permitted to move. The example on the left (*IH3*) is stable, whereas the one on the right (*IH41*) bends. In both examples the key tilings are shown together with thick outlines connecting the tiling vertices.*

simple examples based on linear interpolation are shown in Figure 1. More sophisticated curve morphing techniques such as that of Sederberg et al. [12] might produce more attractive results. Note that Escher's Interpolations relied exclusively on this simple case, or on the variation in which shapes are liberated from the tiling as they become more realistic. Nevertheless, we wish to examine the remaining cases.

When the two tilings are of different isohedral types but have congruent tiling vertices, the aforementioned approach still works. However, Interpolation may produce several incongruent intermediate shapes, violating one of the design principles of parquet deformations. This situation arises when the tiling types have incompatible sets of orientations, causing tiles with different relative orientations to be identified. As shown in Figure 2, we can restore approximate monohedrality by interpolating through an intermediate tiling with straight edges (the so-called *Laves tiling* of the key tilings' isohedral types [6, Chapter 4]). This change reduces Case 2 to two instances of Case 1, though the simplicity of the intermediate tiling can be aesthetically problematic.

Case 3 is easy to carry out temporally. In my previous work on Escherization, I showed how each isohedral tiling type has a simple parameterization that controls the locations of the tiling vertices [8]. Given two tilings of the same type, we can interpolate smoothly from the parameters controlling the vertices in $\mathcal{T}_1$ to those of $\mathcal{T}_2$. We can then interpolate the edge shapes as before. Though continuous, this Interpolation may cause the tiling to undergo an arbitrary affine transformation (as in the case of squares deforming into parallelograms), which does not necessarily make for a very "stable" animation.

The spatial variation of Case 3 is difficult. To draw the Interpolation, we must first lay down an arrangement of tiling vertices that gradually changes from that of $\mathcal{T}_1$ to that of $\mathcal{T}_2$. But even within a single isohedral type, configurations of tiling vertices can change dramatically. The problem is exacerbated by the fact that the Interpolation is done in the same space in which the tiling is drawn. In the temporal case, there is no such interference. One possible solution is to use the underlying correspondence between tiling vertices to linearly interpolate between a tiling vertex's positions in $\mathcal{T}_1$ and $\mathcal{T}_2$. In this case, it makes sense to minimize the global affine transformation between the two sets of tiling vertices, in order to make the line segment connecting any two corresponding vertices as short as possible. Once the tiling vertices are laid out, tile edges can be interpolated as usual. This approach can produce unsatisfactory results because even when the global affine transformation is minimized, the interpolation can still bend and bulge, destroying the clean linear progression found in Huff's deformations (see Figure 3). More work is needed to determine how to align the two tilings in such a way that the interpolation can be done cleanly in a strip.

The fourth case is very much like the third. Because the two tilings have the same topological structure, the Laves tiling with that topology can be expressed in the parameterizations of the isohedral types of both key tilings. This shared tiling can then be used to deduce the correspondence between tiling vertices, from which the previous interpolation methods follow. Note that because we are potentially dealing with incompatible sets of tile orientations, the incongruence problem of Case 2 reappears here. As before, making an explicit transition through the shared Laves tiling would reduce the problem to adjacent instances of Case 3.

The general case is the trickiest; in addition to all the difficulties encountered so far, we must account for a change in the very topology of the tiling. Thus, there can no longer be a clear correspondence between tiling vertices. On the other hand, many of Huff's examples achieve topological transitions without much effort. If we could manually produce suitable Interpolations between the various Laves tilings, we could use these transitions as "gateways" to connect any two isohedral tilings, regardless of topology. Because there are only eleven Laves tiling, a small number of Interpolations are required.

In Figure 4, I propose a set of Interpolations between Laves tilings. Note that these transitions may themselves be concatenated in order to move between two Laves tilings for which a direct transition is not given (though other more direct Interpolations exist that are not shown). This approach unifies all the Laves tiling except for $(4.6.12)$. I conjecture that no smooth transition is possible into or out of that tiling. Fortunately, ignoring $(4.6.12)$ leaves out exactly one isohedral type out of 81 (IH77). The unreachability of that type need not be considered a significant shortcoming.

Until now, we have assumed that when multiple transitions are chained together, the chaining is done through simple concatenation. This approach limits the aesthetic range of Interpolation. In the temporal case, the passage through an intermediate tiling may be continuous, but exhibit a jarring derivative discontinuity. In the spatial case, we would like to pass from tiling $\mathcal{T}_1$ to tiling $\mathcal{T}_2$ in a way that smooths over the intermediate gateway transitions. I hypothesize that in addition to *concatenating* interpolations, we should be able to *compose* them, and have both interpolations occur simultaneously. Any sequence of interpolations could then be composed together, yielding a smooth deformation directly from one tiling to another.

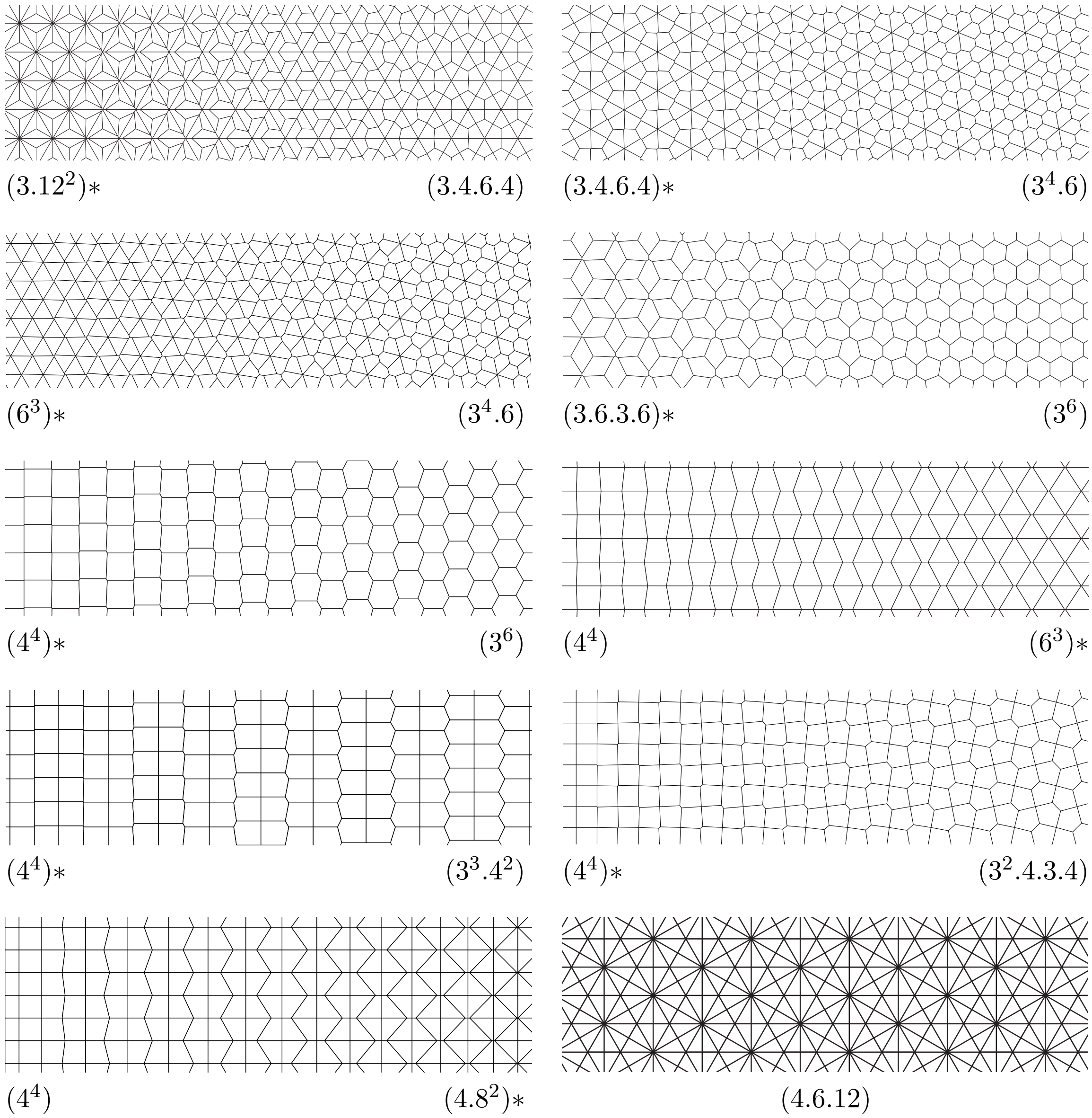

Figure 4 : *A collection of parquet deformations between the Laves tilings. Each deformation starts and ends at a Laves tiling, as marked under the diagram. Each has a topological discontinuity somewhere along its length. These examples all have discontinuities at one endpoint, which is marked with an asterisk. By concatenating or composing these deformations, we should be able to transition between any two Laves tilings other than* $(4.6.12)$*, shown at the bottom right.*

As an analogy, consider the motion of a point along a line segment. If we wish to move from position p_1 to p_2 and then from p_2 to p_3, we might simply concatenate the two trajectories; this new path will exhibit a discontinuous change in direction (and speed, if the segments have different lengths). However, de Casteljau's algorithm for drawing a quadratic Bézier curve with control points p_1, p_2, and p_3 short-circuits the linear trajectory and creates a smooth path that composes the two original segments. It would be interesting to investigate whether there is an analog to de Casteljau's algorithm for this problem of composing tiling interpolations.

4. Conclusions

The main purpose of this paper is to provide a taxonomy of transition types used by Escher. This taxonomy could be used to understand similar work by other artists, or to investigate how we might push beyond these six types with new geometric transitions.

Having presented my analysis of Escher's work, I could not resist speculating on techniques that could be applied to automate Interpolation. Escher's use of Interpolation leads to a deep, fascinating problem in the domain of isohedral tiling. I hope that the suggestions in this paper will stimulate new research in the construction of mathematical metamorphoses.

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