Four Bar Linkages

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Abstract

Four bar linkages (sometimes called three bar linkages) have been investigated by mathematicians, engineers and artists for over three hundred years. A natural extension to three dimensions generates a family of aesthetically interesting surfaces.

The Lemniscate

The plane curve known as the lemniscate was first described by Jakob Bernouilli in 1694 (in *Acta Eruditorium*). There are several elegant ways to construct this curve [1], perhaps the best known is by means of a four-bar linkage, known as Watt's linkage. This was widely used in steam-driven beam engines to generate approximate straight line motion [2]. Two bars of unit length pivot around points that are fixed √2 apart (by a notional third bar). They are joined by a fourth bar, length √2, and the mid-point of this bar moves on a lemniscate, while its ends describe unit circles (figure 1). John Sharp has written an article that describes some less well-known properties of this curve [3].

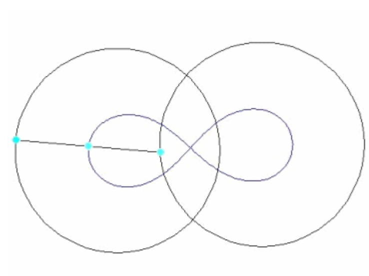
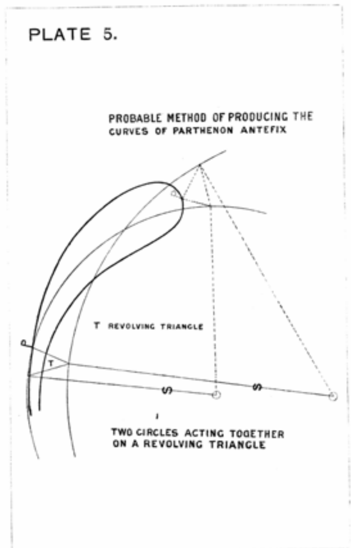


Figure 1: *The lemniscate of Bernouilli as a locus of a point on a four bar linkage*.

More General Curves

It is easy to see that this arrangement can actually generate a five parameter (one is a scale factor) family of curves by altering the lengths of the four bars and the position of the drawing point on the fourth bar (possibly extended). Computer geometry packages provide a convenient way to investigate these curves, and there are several interactive animations available on the internet, for example [4].

The drawing point need not lie on the fourth bar but could be fixed to it some distance away, an idea that was used by Daniel Wood, an architectural mason, at the end of the nineteenth century [5]. Figure 2 shows one of the plates from his book. The pivoting bars are strings, each labelled S, the fourth bar is a triangle, labelled T, and the drawing point is labelled P.The idea was developed further by Richard S. Hartenberg, a professor of mechanical engineering, who co-authored a book on linkages [6], and published a paper in *Leonardo* describing how to build a machine to produce a range of interesting curves [7]. Figure 3 shows a photograph of the machine with nineteen curves produced by it.



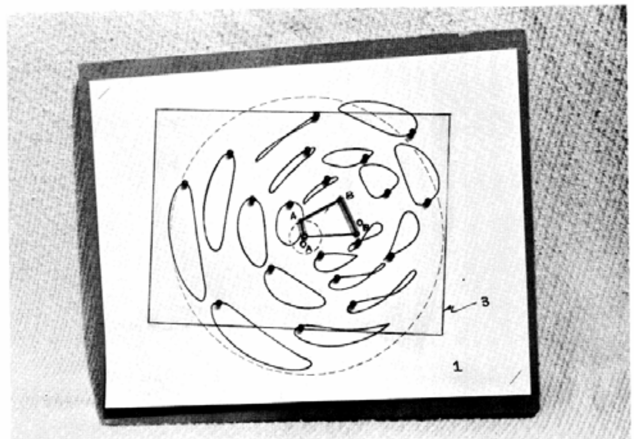


Figure 2: *Wood's method of drawing curves*. **Figure 3:** *Hartenberg's machine (from* Leonardo).

This linkage was analysed in a series of papers published by the London Mathematical Society [8], and an extensive collection of the curves it can produce was published in 1951 [9].

Extension to Three Dimensions

Reverting to the simpler situation when the drawing point lies on the fourth bar, and allowing one of the pivoted bars to move out of the plane, so it describes a sphere rather than a circle, generates a surface, rather than a curve. (Allowing both bars to move on spheres would need four dimensions to represent the locus properly.) The surface consists of circular sections, since for any position of the fourth bar in the 2-D arrangement one end is on the circle but the other moves on the sphere, so the bar describes a cone in 3-D, with the drawing point describing a circle. Figure 4, generated (as are all the following figures) using VRML2 with calculations in a Javascript Script node, shows the surface that corresponds to the lemniscate.

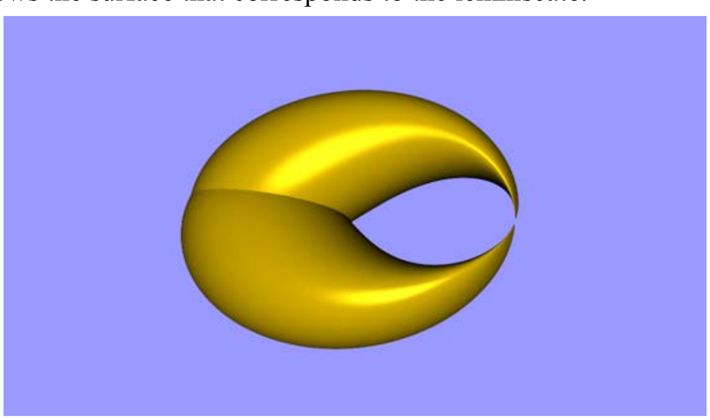
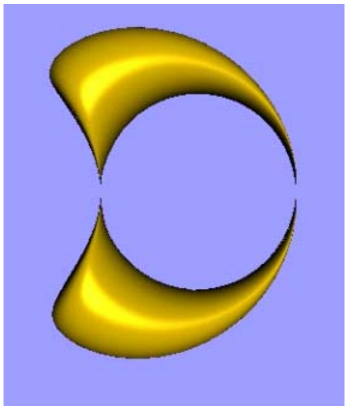


Figure 4: *The surface corresponding to a lemniscate.*

There is a surprising variety of surfaces produced just by varying the position of the drawing point on its bar, keeping all other parameters the same. Call the drawing point P, the end of the bar moving on the circle C, and the end moving on the sphere S. Figures 5-7 illustrate some possibilities



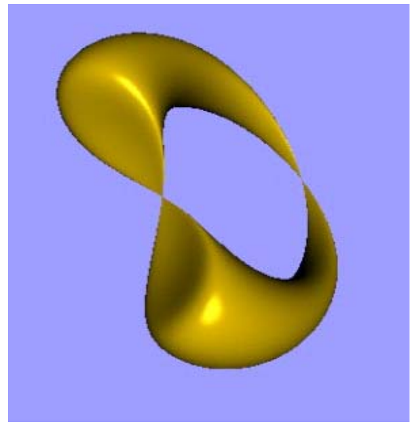


Figure 5: *Two views of the surface when the drawing point, P, is some distance beyond C, on the circle.*

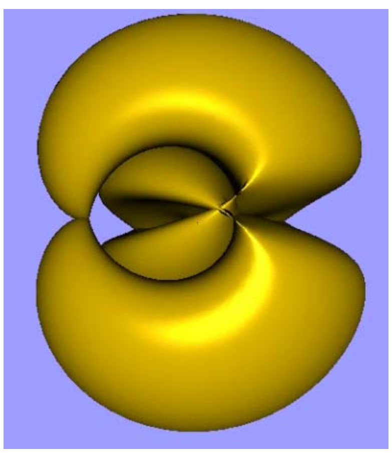


Figure 6: *The surface when P is beyond S.*

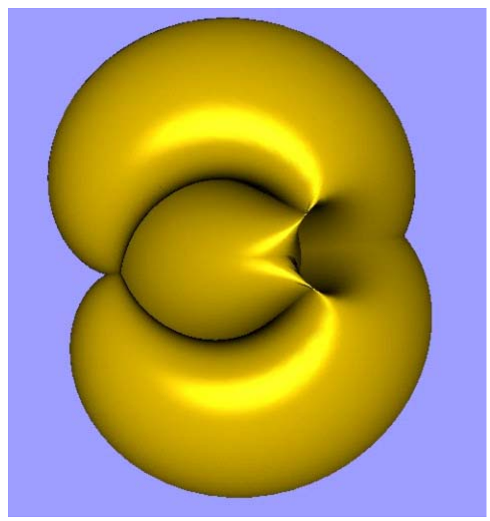
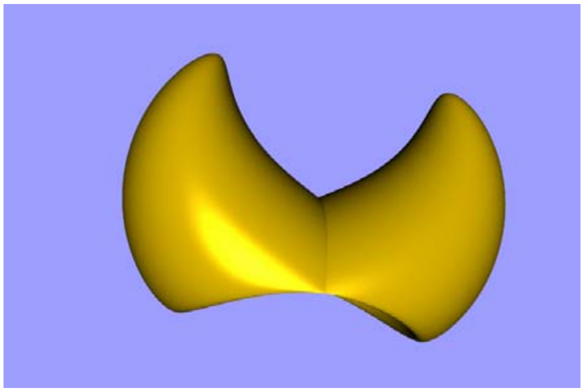


Figure 7: *The surface when P is further beyond S.*

Of course the shapes of these surfaces are determined by four parameters, so there is a wide range of possibilities. Figure 8 shows the effect of reducing the length of the fourth bar, and figure 9 has a longer fourth bar with P beyond S, rather than at the mid-point. Figures 10 and 11 illustrate some others.

The table gives the parameters for the various figures (radius of circle, radius of sphere, length of fourth bar, position of drawing point: sphere is 1, circle is 0). The distance between the fixed pivots is set at 1.

figure	r circle	r sphere	length	position
4	0.7071	0.7071	1	0.5
5	0.7071	0.7071	1	-0.5
6	0.7071	0.7071	1	1.8
7	0.7071	0.7071	1	2
8	0.7071	0.7071	0.7	0.5
9	0.7071	0.7071	1.4	1.6
10	1	1	1	1.1
11	1	1	1.001	1.7



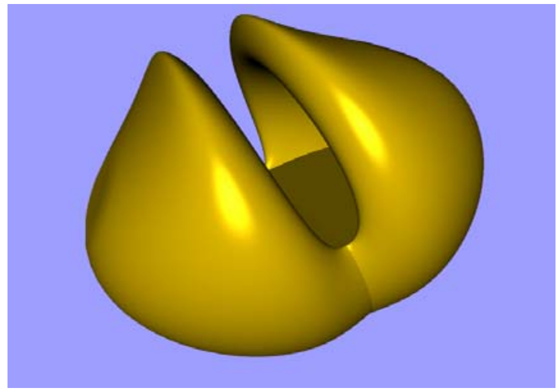
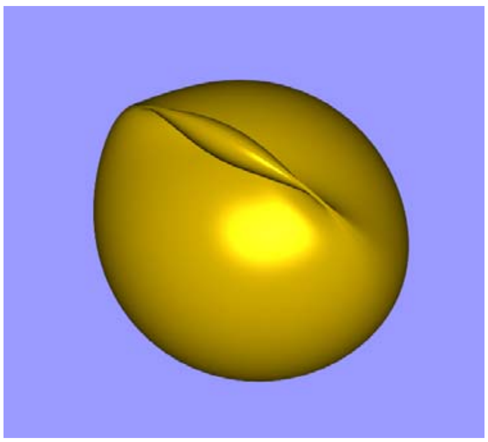
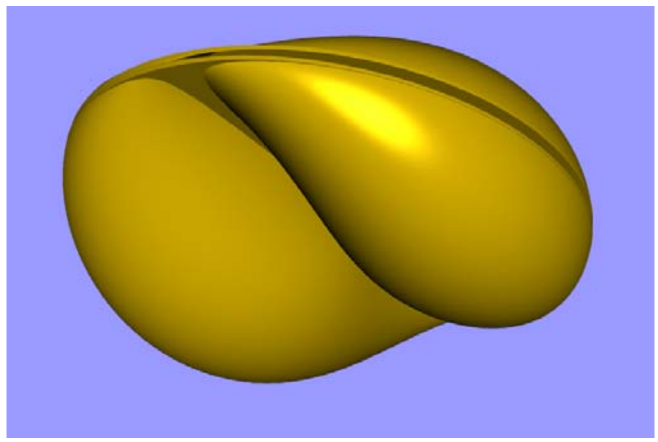


Figure 8: *Shortening the fourth bar.* **Figure 9:** *Lengthening it, with P beyond S.*





Figures 10 and 11: *Two more surfaces generated by changing all four parameters.*

This brief survey has given some indication of the variety of forms that can be produced from the motion of three bars connected to two fixed points. Of course other linkages are known, but the possibilities seem to have been investigated only in two dimensions, or possibly in three dimensions with the linkage constrained to execute a space curve. It is likely that many of them can generate visually interesting forms, and much remains to be discovered.

References

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[4] http://enriques.mathematik.uni-mainz.de/intgeo/threeBarLinkage.html (accessed Dec.2007)

[5] Wood, D., *A Handbook of the Greek Method*, Whiting & Co., London, 1889.

[6] Hartenberg, R.S., and Denavit J., *Kinematic Synthesis of Linkages*, McGraw-Hill, 1964.

[7] Hartenberg, R.S., Paths by Coupler for Kinetic Art, *Leonardo*, Vol.4 (Spring 1971), pp.125-8, reprinted in *Kinetic Art: Theory and Practice*, ed. Malina, F.J., Dover, 1974

[8] Cited in Morley and Morley, *Inversive Geometry*, G.Bell and Sons Ltd., London, 1933, p.184.

[9] Hrones, J.A., and Nelson, G.L., *Analysis of the Four Bar Linkage*, MIT, 1951 (also John Wiley, 1951).