Perspective Drawings of Lattices

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Abstract

We consider several amusing aspects of drawings of lattices; in particular, drawings of a lattice of flat objects parallel to a viewing plane are periodic.

Figure 1 is a drawing, in perspective of disks, parallel to the viewing plane, arranged at the corners of a cubic lattice, which begs the question: *What is going on here?*

The most surprising aspect of this image is its periodicity; in fact, if we made a rendering of a perfect 180 degree field of vision, the pattern would repeat infinitely, covering the entire plane. This strains our credulity and we have to ask ourselves: *Does a lattice of disks really look just this way?*

In this age, we see realistic renderings– photographs, videos, etc.—practically every waking moment of our lives. We have become so versed in reading such images that it is almost impossible for us to remember that any accurate rendering in perspective, from a camera or from an artist's hand, is meant to be viewed from a very specific vantage point.

The image in Figure 1 really is only accurate when viewed from a precise spot: directly above the center of the image, exactly one-square's-width away from the page. Take a look! (It might be easier to see if the image is enlarged first)

If you can manage to put your eye in the correct spot, and still see the image, then you will see that the apparent periodicity of the rendering vanishes—the *drawing* is periodic, but the *apparent image*, the actual appearance of the image, is not. It is increasingly foreshortened away from the center of the image, and the disks appear (as they should) to be ellipses.

In Figure 3, we see some of the effects of moving the eye relative to the plane of the rendering and the disks being rendered. An interactive version of this figure can be examined at [3] and short animations of similar effects are at [2].

But why is the drawing itself periodic? At first glance, this seems counterintuitive. Yet it is really quite simple, as illustrated in Figure 4; a two-dimensional array of pegs is shown, relative to an eye and a plane on which the pegs are projected. Overlaying this, we see a drawing that appears to the eye just the same as the pegs do. In effect we are layering multiple copies of a particular pattern, each scaled down by an integer factor. Indeed, the proof is elementary:

Lemma 1 *Let $X \subset \mathbb{R}^3$ so that for all $(x,y,z) \in X$, z is a non-zero integer. Suppose there exists a vector $v = (a,b,0)$ so that $X + v = X$. Let $X' = \{(x/z, y/z, 1) \mid (x,y,z) \in X\}$. Then $X' + v = X'$.*

In other words, if the points in X have only non-zero integer valued z-coordinates, and if X is invariant when shifted by v (and so is periodic in the v-direction), then X', the central projection of X onto the plane $z = 1$, is also invariant when shifted by v. The proof is trivial:

Proof Let X and v be as above. Then for any $(x', y', 1) \in X'$, there exists an integer $n \neq 0$ so that $(nx', ny', n) \in X$. But then $(nx', ny', n) + n(a,b,0) = (n(x'+a), n(y'+b), n)$ also lies in X and $(x'+a, y'+b, 1) = (x', y', 1) + v$ lies in X'. $\square$

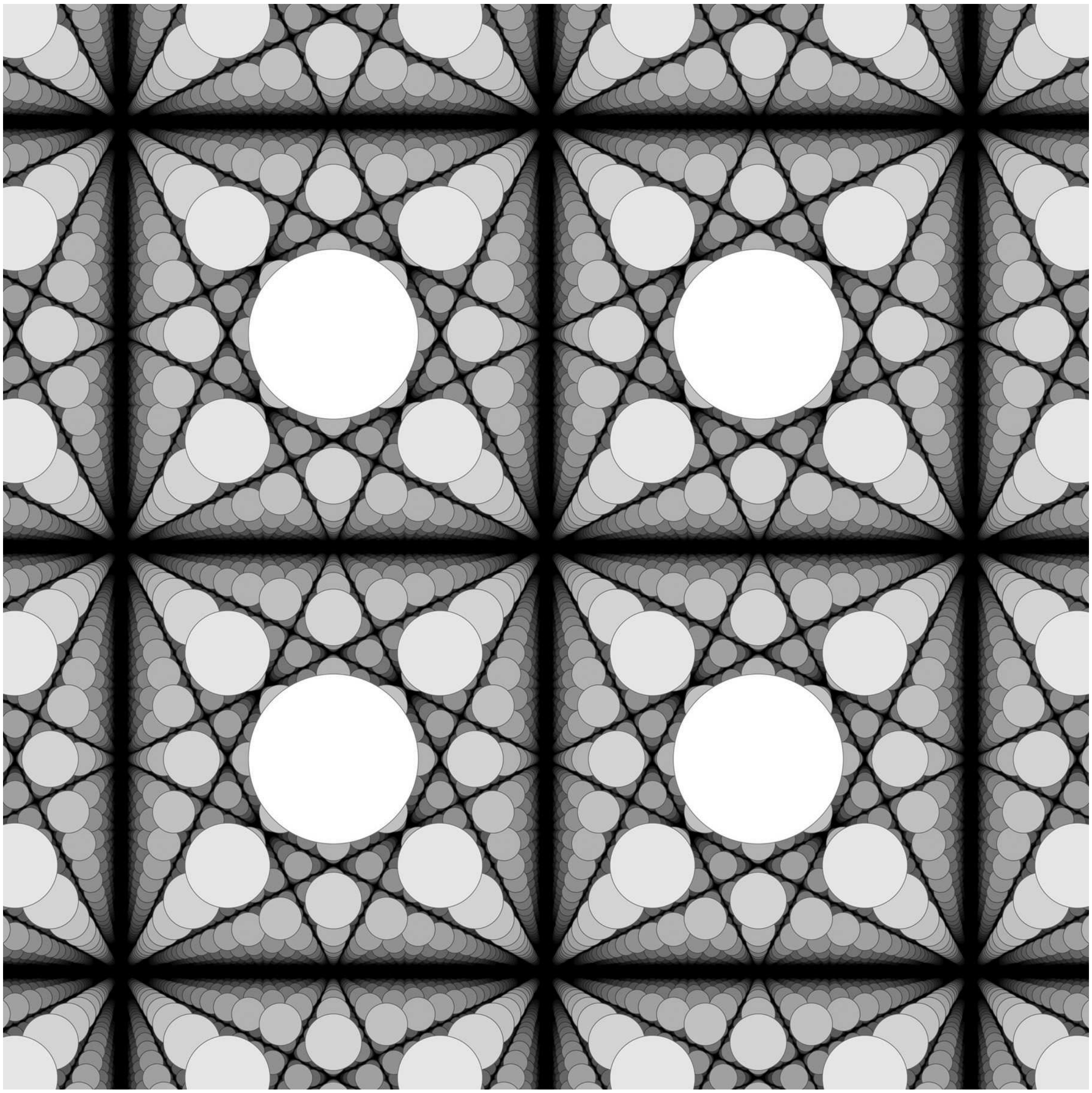

Figure 1: In order to view this rendering of a cubic lattice of flat disks, parallel to the viewing plane, your eye should be exactly one square's-width (approximately one third of the width of this drawing) above the center of the image– of course it will help to enlarge the drawing!

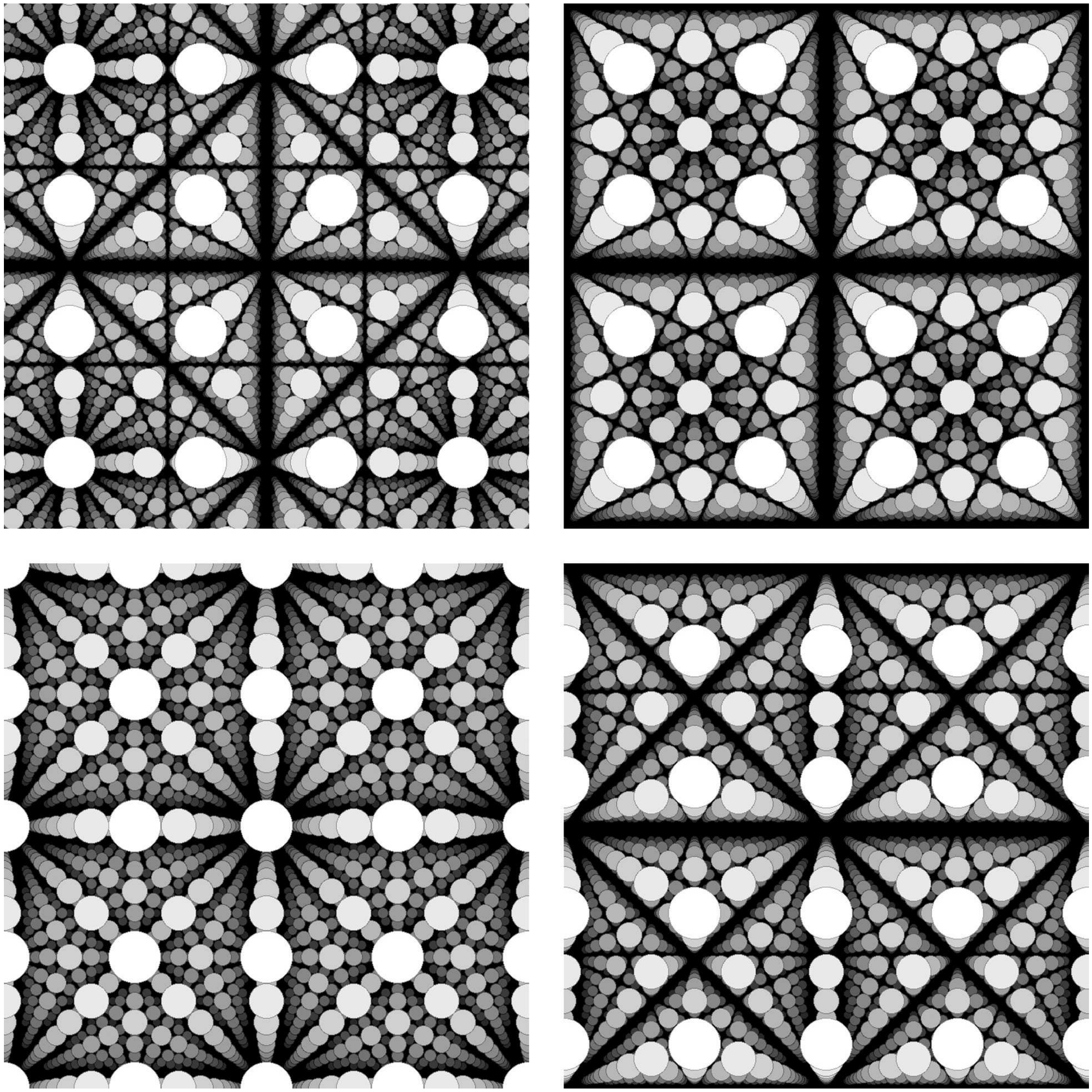

Figure 2: In each of these drawings, the plane of the image remains the same, relative to the lattice of disks, but the position of the eye has changed. In the upper figures, the eye is further away than in Figure 1: the eye should be 1 1/2 times as far away in the upper left image, and twice as far away in the upper right. In the lower figures, the eye is the same distance from the image plane, but the cubic lattice has been moved laterally.

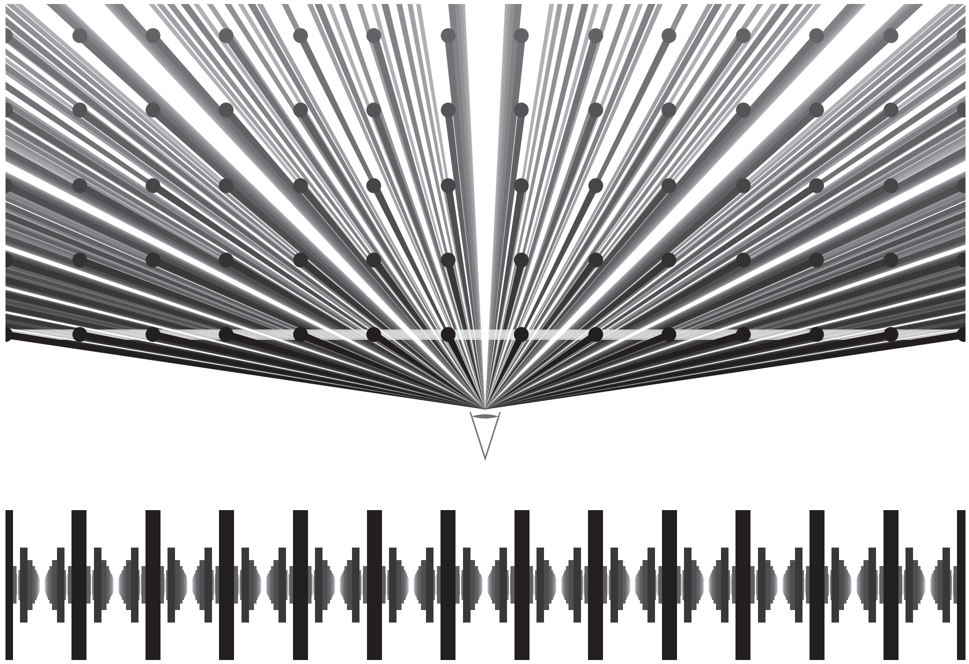

Figure 3: Why the images are periodic; here we see the image of a lattice of pegs. The gaps correspond to sight-lines with integer slopes.

Despite the mathematical triviality of this observation, we find the phenomenon artistically surprising and worth this discussion.

These periodic perspective drawings are of *flat* objects, lying within planes parallel to the image plane. Drawings of objects that have some thickness become increasing skewed the more obliquely an object is viewed.

In Figure 4 we see a cubic lattice of spheres; again, the correct vantage point is one square's-width away from the center of the drawing. When viewed from this vantage point, all of the ellipses in the drawing appear to be circles, as spheres should.[1]

It is quite amusing to consider renderings on non-planar surfaces; Dick Termes has explored this to great effect in his spherical paintings [4]. In Figure 5 we show how a cubic lattice of spheres (not disks) would appear when projected directly onto a sphere. That is, if you were to place your eye in the exact center of such a spherical image, looking out, you would see a cubic lattice of spheres in perfect perspective. In the first image, the cubic lattice is aligned with the eye—the center of the spherical image is a point in the lattice. In the second, the lattice has been shifted, so that the eye is in the center of a cube.

We find these images quite pretty, and would like to know more about these arrangements of circles on these spherical images.

Finally, what about the way that lattices appear on an spherical retina? Strikingly, if you compare the flat images we are used to—those that might be produced on a photographic plate—with those on an idealized, spherical retina, the flat image is a stereographic projection of a retinal image. Or, to put it the other way around, the images on our retinas are stereographic inverses of the kinds of planar renderings we are used to!

[1] Since it took us some trouble to work out, we pause to give the measurements of these ellipses: suppose we render a sphere of radius r, at position (i, j, k) in space, as viewed from the origin, on a drawing on the plane $z = d$; then on this drawing, the sphere appears as an ellipse, with center $\frac{dk}{k^2-r^2}(i, j)$; major radius $\frac{dk\sqrt{R^2-r^2}}{k^2-r^2}$; minor radius $\frac{dr}{\sqrt{k^2-r^2}}$; and eccentricity $\frac{\sqrt{i^2+j^2}}{\sqrt{R^2-r^2}}$, with the major axis aligned towards the origin.

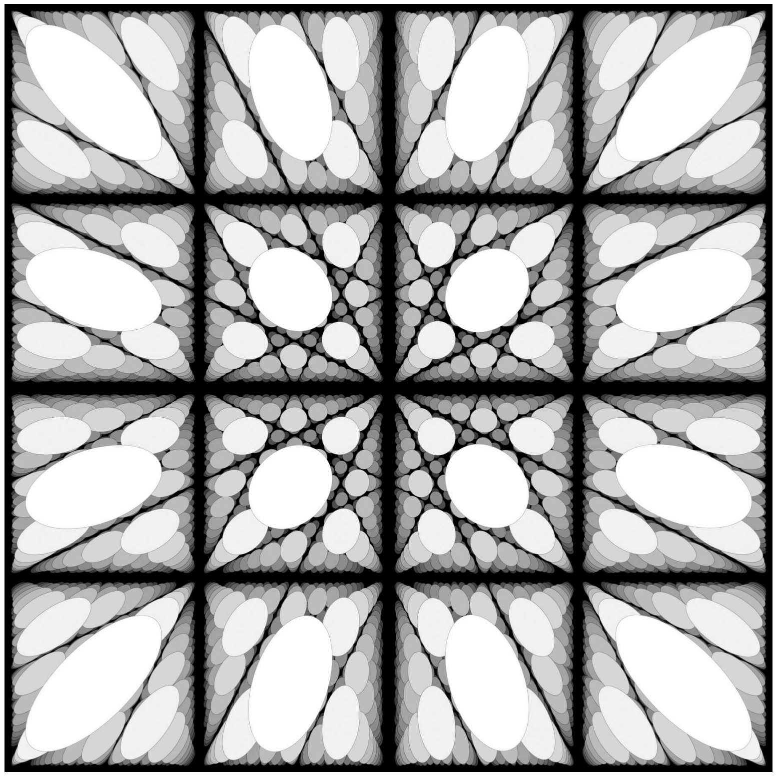

Figure 4: This is an accurate rendering of a lattice of spheres; when viewed one square's-width directly above the center of the drawing, the ellipses appear to be circular—the projected images of spheres.

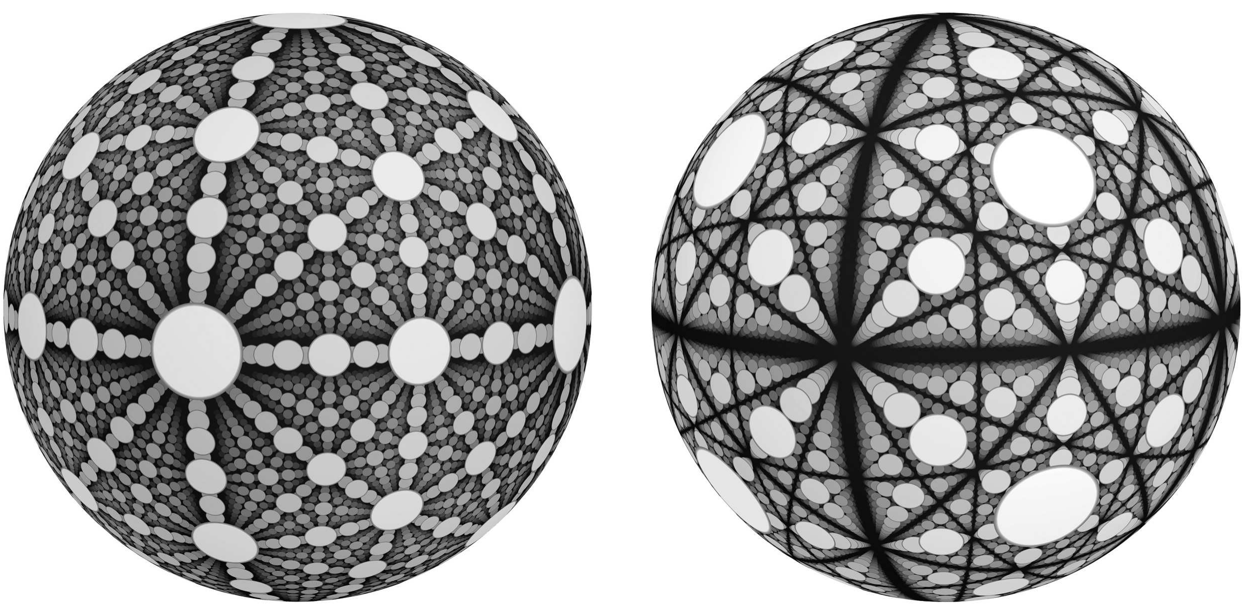

Figure 5: Spherical renderings of a cubic lattice of spheres; these renderings should be viewed from their centers, in the interior looking outward.

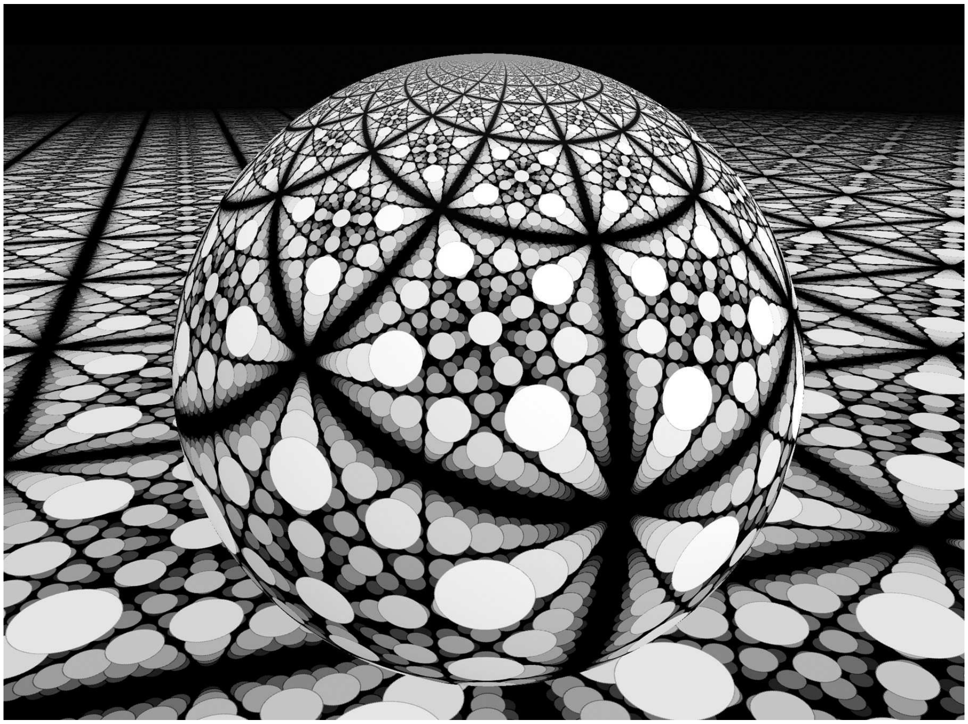

Figure 6: The image on a retina is the stereographic inverse of a planar image.

Mathematically, this is trivial—follow a ray through a pinhole, into the eye, through the retina, and down to a plane behind the eye— but we find it quite surprising and strange to contemplate. In our final figure, we illustrate two renderings of a cubic lattice of disks, one on a retina, and one on a plane; the planar image is a stereographic projection of the retinal image.

References

[1] J.H. Conway, H. Burgiel, C. Goodman-Strauss, *The Symmetries of Things*, AK Peters (2008).

[2] H. David, *http://www.tabletoptelephone.com/ ˜hopspage/Harmonic.html; http://www.tabletoptelephone.com/˜hopspage/Sineanim.html*

[3] C. Goodman-Strauss *http://demonstrations.wolfram.com/LatticesInPerspective*

[4] R. Termes, *http://termespheres.com*