The Mathematics of Mitering and Its Artful Application

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Abstract

We give a systematic presentation of the mathematics behind the classic *miter joint* and variants, like the *skew miter joint* and the *(skew) fold joint*. The latter is especially useful for connecting strips at an angle. We also address the problems that arise from constructing a closed 3D path from beams by using miter joints all the way round. We illustrate the possibilities with artwork making use of various miter joints.

1 Introduction

The miter joint is well-known in the Arts, if only as a way of making fine frames for pictures and paintings. In its everyday application, a common problem with miter joints occurs when cutting a baseboard for walls meeting at an angle other than exactly 90 degrees. However, there is much more to the miter joint than meets the eye. In this paper, we will explore variations and related mathematical challenges, and show some artwork that this provoked.

In Section 2 we introduce the problem domain and its terminology. A systematic mathematical treatment is presented in Section 3. Section 4 shows some artwork based on various miter joints. We conclude the paper in Section 5 with some pointers to further work.

2 Problem Domain and Terminology

We will now describe how we encountered new problems related to the miter joint. To avoid misunderstandings, we first introduce some terminology.

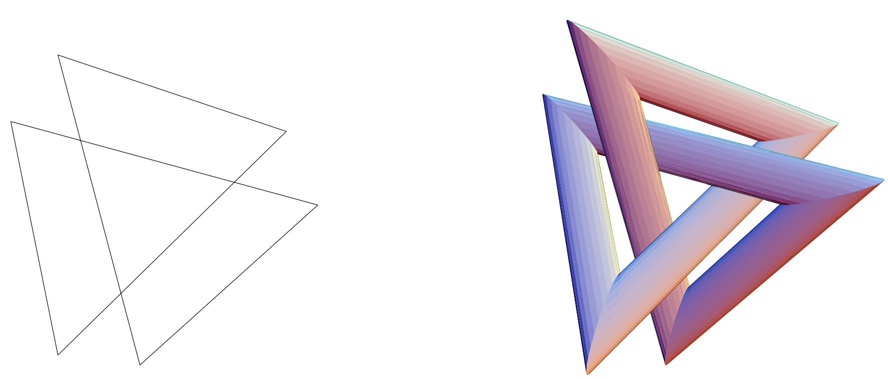

Figure 1: *Polygon knot with six edges (left) and thickened with circular cylinders (right)*

2.1 Cylinders, single and double beveling, planar and spatial mitering

Let $\mathscr{K}$ be a one-dimensional curve in space, having finite length. Such a curve is infinitely thin and thus difficult to realize faithfully in the physical world. To make the curve more tangible, one can search for ways to *thicken* it.

Let us assume that $\mathscr{K}$ is a finite chain of line segments. The knotted hexagon with sixfold symmetry depicted left in Figure 1 serves as an example. Hard to see, isn't it?

A simple thickening is obtained by blowing up the line segments into **circular cylinders** with diameter d. The original line segments are the *center lines* of these cylinders. Some care is needed where the line segments meet. Cutting the cylinders along the interior bisector plane of the angle where they meet, results in an elliptic **cut face**. The cylinders of two neighboring line segments fit together smoothly at those ellipses, as can be seen on the right in Figure 1. Of course, it is a practical matter to restrict d such that non-neighboring cylinders will not intersect. In general, determining the maximal value for d is a challenge (which we will not address in this article).

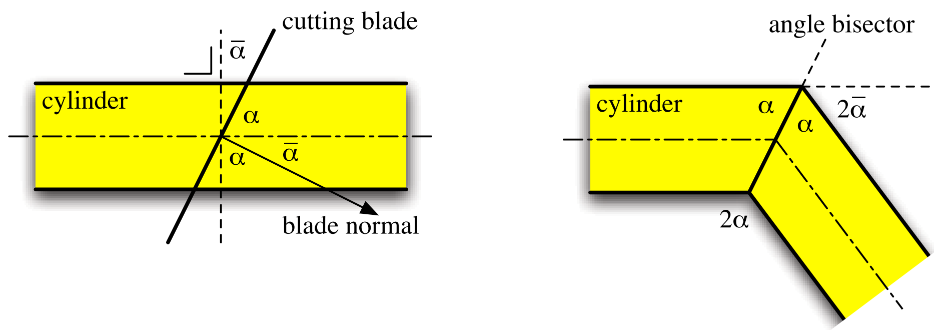

Figure 2: *Top view of cylinder being beveled (left) and resulting miter joint (right)*

Cutting off a cylinder at an angle is called **beveling** (see Figure 2). This is usually done in a **miter box**. The cylinder is clamped horizontally inside the box. The cutting blade moves vertically down, and can rotate (yaw) around the vertical axis to set the **bevel angle** α between blade and center line of the beam, or equivalently between the blade normal (perpendicular to the blade) and the plane perpendicular to the beam. The complement of α is the angle $\overline{\alpha}$ between blade normal and center line, or equivalently between blade and plane perpendicular to the beam. The joint angle after mitering equals 2α, that is, twice the bevel angle. Originally, the term miter joint applied only to right-angled joints, with $\alpha = \overline{\alpha} = 45°$. The classic picture frame is an example.

The cylinder can be rotated in the miter box around its center line. Because of the full rotational symmetry of a cylinder, this rotation has no effect when beveling the first end. When beveling the second end, however, the cylinder must be rotated over an appropriate angle β in order to follow the planned path. Referring to Figure 3 (left and middle), consider a line segment ℓ to be thickened, connecting to two other line segments ℓ_- and ℓ_+. Segments ℓ and ℓ_- span a plane, and so do ℓ and ℓ_+. The angle between these two planes is the desired rotation angle β. In the case of $\beta = 0°$, the three segments ℓ_-, ℓ, and ℓ_+ lie in one plane (are coplanar), and we call this **planar mitering**. For $0° < \beta < 180°$ we speak of **spatial mitering**. In practice, the rotation of a circular cylinder is not so easy to carry out accurately, because it lacks a clear reference position for measuring angles due to its roundness.

Note that our distinction between planar and spatial mitering involves *both* ends of the cylinder. Some authors use the terms planar and spatial mitering differently, viz. when beveling a single end [1]. This confusion can be explained as follows. Instead of rotating the cylinder (around its center line), one can equivalently rotate (roll) the blade away from its vertical position and adjust its yaw angle α appropriately. This could be called **double beveling**, in contrast to **single beveling**, where the blade remains vertical and has only *one* degree of freedom as opposed to two. Whether double beveling of one end yields a spatial

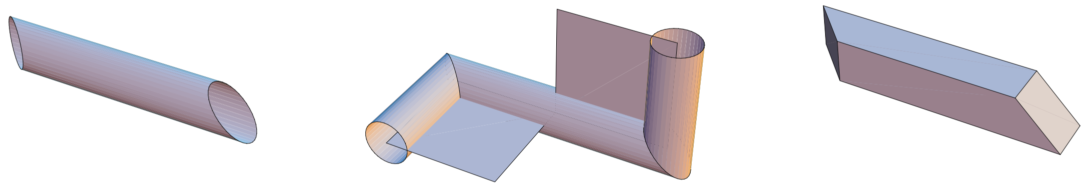

Figure 3: *Cylinder with two beveled ends (left), spatial mitering with angle-spanning planes at $\beta = 90°$ (middle), piece with square cross section beveled at both ends resulting in rectangular cut faces (right)*

structure (with non-coplanar segments) depends, however, also on what happens at the *other* end. Single and double beveling say something about the working method, but not about the planar or spatial effect of multiple joints.

2.2 Polygon as Cross Section

The circular cross section used above for thickening can be replaced by any other shape. Common choices are simple polygons, like squares, rectangles, and even rhombuses, parallelograms, or various triangles. In mathematics, a cylinder is any set of parallel lines, not necessarily circular in cross section. We will also use the term **beam** to refer to a cylinder with polygonal cross section. These cross sections work almost the same way as circles. The main difference occurs at the joints. It is more pleasing if the edges of both beams connect properly across the joint, as is usually the case in a picture frame. We say that the beams **match**, to distinguish it from the situation where edges do not connect properly.

First consider a *square* cross section; that is, cutting the cylinder perpendicular to its center line ($\alpha = 90°$) yields a square cut face. When beveled at an angle $0° < \alpha < 90°$, the resulting cut face is in general a *parallelogram*: opposite sides remain parallel, regardless of cutting angles α and β. In the special case where the beam lies flush in the miter box, beveling with a vertical blade yields a *rectangular* cross section (see on the right in Figure 3).

Just like an ellipse, a parallelogram has two (rotational) symmetries. Hence, after beveling the square beam, the two pieces can be matched in two ways. In the first way, the pieces remain together as before cutting, which is uninteresting (why cut at all?). The second way is obtained by rotating one beam over 180° around the perpendicular center line of the parallelogram. This way we obtain an angle between the beams, such that the common cut face *bisects* the angle and is perpendicular to the plane spanned by the two beams.

A practical advantage of using a polygon as cross section is that beveling is easier when the beam needs to be rotated around its center line over angle $\beta > 0°$. The edges on the beam can serve as a natural reference for such rotation. Ideally, the beam lies flush in the miter box. In the case of a square beam, that leaves only two interesting rotation angles: $\beta = 0°$ and $\beta = 90°$. By the way, a disadvantage of polygons as cross section is that it is even harder to determine at what thickness d non-neighboring beams start to intersect. The complicating factor is that now the rotational 'phase' of the beams matters, and not just the distance between their center lines.

2.3 Surprising Twist

One might think that, by using matched miter joints, every spatial polygon $\mathscr{K}$ can be thickened to polygonal cylinders having the same cross section. For *planar* polygons $\mathscr{K}$, like the picture frame, this is indeed the case. However, for *nonplanar* polygons $\mathscr{K}$, like the hexagonal knot of Figure 1, it generally does not work out. The game starts to become mathematically interesting, because of a surprising twist.

We pick a line segment to start the thickening process. Blow up one *half* of it to, say, a *square* cylinder. Cut it off at the next vertex along the interior bisector plane. The resulting cut face determines the matched thickening of the next line segment. Finding appropriate values of α and β and carrying out the cutting accordingly, is a matter of craftsmanship.

Continue along the entire polygon, and then see what happens after the last vertex, when the remaining half of the initial line segment is thickened. Figure 4 (left) shows that, in the case of our hexagonal knot, the edges fail to match. In general, the edges can be made to match everywhere, *except* possibly halfway the initial segment. There, a **twist** may occur. The mitering does not match properly.

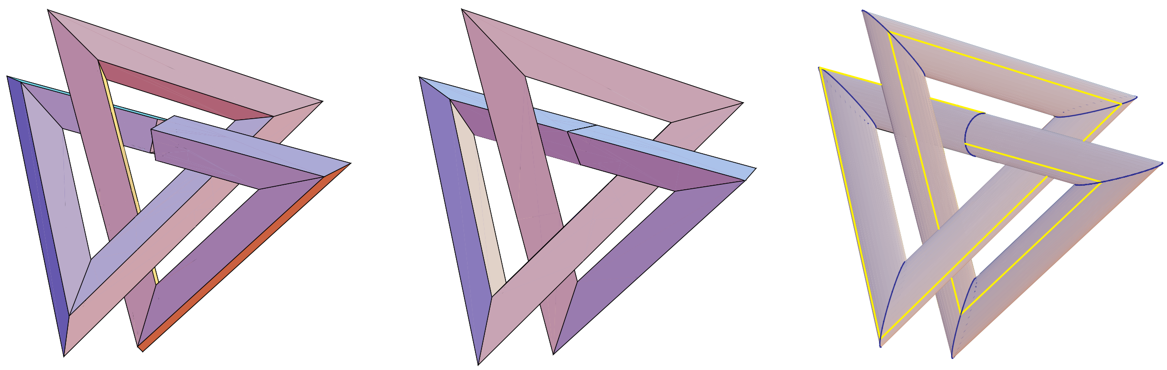

Figure 4: *Square cross section fails to match (left), equitriangular cross section does match (middle), and circular cross section with yellow seam revealing the* $120°$ *rotation (right)*

Of course, one can try and use a different polygonal cross section. It does not have to be a square. Figure 4 (middle) uses an equilateral triangle as cross section, which does result in a completely matched mitering. Actually, the positions of the vertices of the hexagonal knot were *designed* to yield a matched mitering for this triangular cross section! Note that the resulting object has only two symmetries, whereas the knotted hexagon has six. In general, the thickened object inherits a *subgroup* of the symmetries of the original polygon.

Because of the way we have set up the thickening process, the two ends of the 'cylindrified' polygonal space walk $\mathscr{K}$ meet in the middle of the initial segment. The polygonal cross sections at these two ends are *congruent*, but they need not coincide. One end may be rotated with respect to the other: the twist. The mitering matches if, and only if, the amount of rotation happens to be a *symmetry* of the polygonal cross section. Figure 4 (right) shows again the hexagonal knot thickened to circular cylinders. But this time a *seam* is drawn on the cylinders. This seam runs parallel to the center line, and matches nicely at all joints. Again we see that it does not match all the way round. The amount of rotation is readily visible. In the case of this particular hexagonal knot, the amount of rotation was designed to equal $120°$.

We have the following **Miter Joint Rotation Invariance Theorem**: The total amount of rotation does *not* depend on

- the choice of initial segment to start the process;
- how much the cross section is initially rotated around the center line;
- the shape of the cross section.

Thus, if the mitering does not match, then rotating the cross sections will not improve the situation. And if all miters happen to match, then after rotating all cross sections 'synchronously', they still all match. This freedom can be exploited to obtain a configuration which can be put down stably on the side of a beam.

Given a polygon $\mathscr{K}$ in space, it is a mathematical challenge to determine its amount of rotation, also referred to as its **torsion**. When designing objects, the *inverse problem* is, however, more relevant: How to locate the vertices in space, such that the resulting polygonal curve has the desired properties. These properties include such things as symmetries, knot class, and aesthetic appeal, but also a suitable torsion, which needs to be a symmetry of the beam's cross section. The inverse problem is usually more difficult than just determining the torsion. In most cases, it cannot be solved analytically and requires that one resorts to numerical approximation. The locations of the vertices offer many degrees of freedom, which also makes solving the inverse problems computationally costly.

2.4 Strip as Cross Section

The polygonal cross section can degenerate into a line segment. In that case, the beam becomes infinitely thin and turns into a (2-dimensional) **strip**. Besides via the regular miter joint, strips can be 'joined' at an angle in another way, viz. by **folding**. When folding a strip, the fold line lies in the *exterior angle bisector plane* of the extended strips (see Figure 5, left). To our surprise, the PostScript Language offers a miter joint to join two line segments, and also beveled and rounded joints, but not a fold joint.

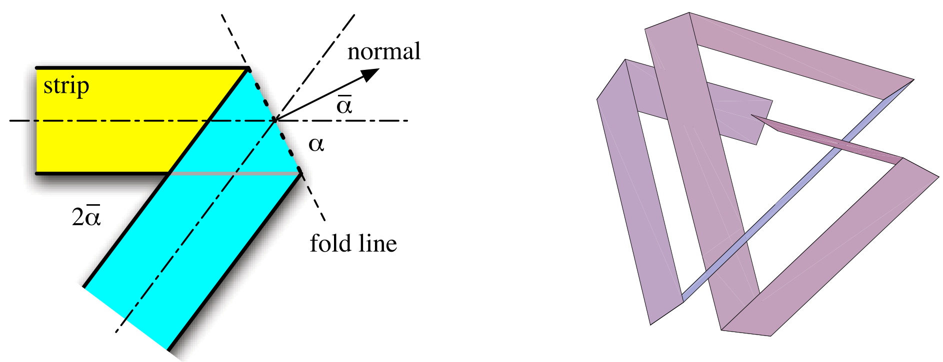

Figure 5: *Fold joint of two equally wide strips (left); hexagonal knot thickened to a folded strip, which fails to match (right)*

Folding a strip along a given spatial polygon $\mathscr{K}$ also suffers from the torsion problem (see Figure 5, right), but in a different way. The fold joints alternate in contributing in a positive and a negative sense to the torsion, whereas miter joints all contribute in the same sense. A fold joints applies a reflection to the torsion. When you rotate a strip along its center line, the next strip connected via a fold joint will rotate in the *opposite* direction. Beams connected via miter joints will rotate in the same direction.

We have the following **Even Fold Rotation Invariance Theorem**, similar to miter joints. If polygon $\mathscr{K}$ has an *even* number of vertices, then the total amount of torsion across all fold joints does *not* depend on the choice of initial segment to start the process, and on how much the strip is initially rotated around its center line. The ends that meet in the middle of the initial segment will rotate in the same direction, maintaining their angular distance. However, in case of an *odd* number of vertices, the ends will rotate in *opposite* direction. Thus we have the **Odd Fold Matching Theorem**: If polygon $\mathscr{K}$ has an *odd* number of vertices, then there exists a rotation of the initial segment for which all fold joints will match. In fact, there exist two distinct ways: one results in a two-sided strip, the other in a one-sided strip with a Möbius twist.

3 Mathematical Treatment

In this section, we analyze miter joints mathematically. The main result is a complete characterization of all configurations of two beams with the *same* cross section to be connected by a *matched* miter joint.

3.1 From Cross Section to Cut Face

We cut a beam having polygonal cross section S to make a miter joint. The shape of the resulting cut face F is obtained by an **oblique parallel projection** of S in the direction of the beam's center line onto the cut plane. For a regular miter joint, the cut plane is the interior bisector of the angle to be mitered. The orientation of the cut plane is described by the angles α and β as explained in Section 2.1. The shape of F is again a polygon, having the same number of vertices as S.

In Figure 6 we have collected a set of beams, all having the same rectangular cross section with aspect ratio $1:\sqrt{2}$, but with varying cut angles. The shape of the cut face varies correspondingly, including various parallelograms and rectangles, a square, and by interpolation a rhombus. The square is in the middle row on the left with $\alpha = 45°$, thus the resulting miter joint has an angle of 90°. A parallelogram has two symmetries, the square has eight. These extra square symmetries suggest that you can also join the beam to a beam with the same cross section in a way *different* from a regular miter joint. We get back to that later on.

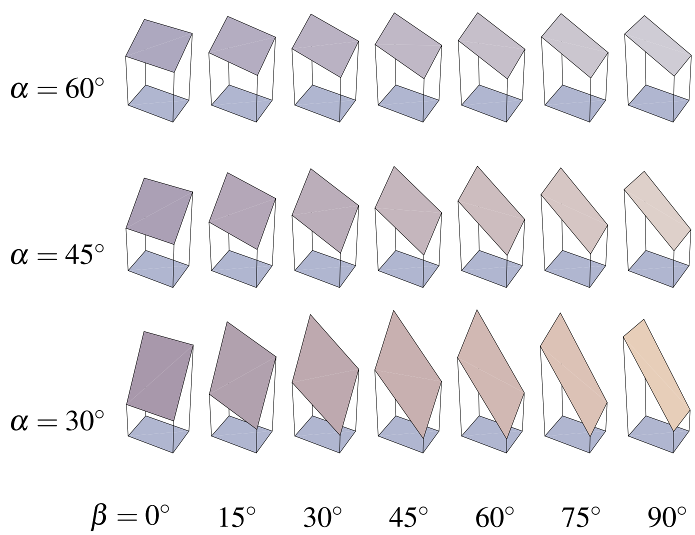

Figure 6: *Beam with rectangular* $1:\sqrt{2}$ *cross section cut at various angles*

3.2 From Cut Face to Cross Section

Instead of focusing on the cross section S of the beam and deduce the shape of the cut face F, one can also start with the cut face. This is, for instance, useful to impose certain symmetry properties on F. The cross section is then determined by the angles at the cut face.

Given the shape of the cut face F, one obtains the shape of the cross section S by a **perpendicular parallel projection** of F in the direction of the beam's center line onto a plane perpendicular to the beam. When using cross section S to thicken the entire closed polygonal curve $\mathscr{K}$, one can determine the torsion by repeating the projection process all the way round. It alternates from cross section to cut face at a vertex and from cut face again to cross section. After traversing the entire curve, the torsion is obtained by tracing a particular vertex of S and determining the angular distance between its initial and final position.

Let us forget about the prescribed angle at a joint, and investigate the shapes that the cross section can take on as the angle varies. In Figure 7 we have collected a set of beams, all having the same $1:\sqrt{2}$ rectangular *cut face*, but approaching the cut at different angles. The cross section varies accordingly.

Now a question imposes itself. If we pick one cross section from the set, at what *other* angles does that *same* cross section appear as well? For a miter joint we match the beam with its mirror image, approaching

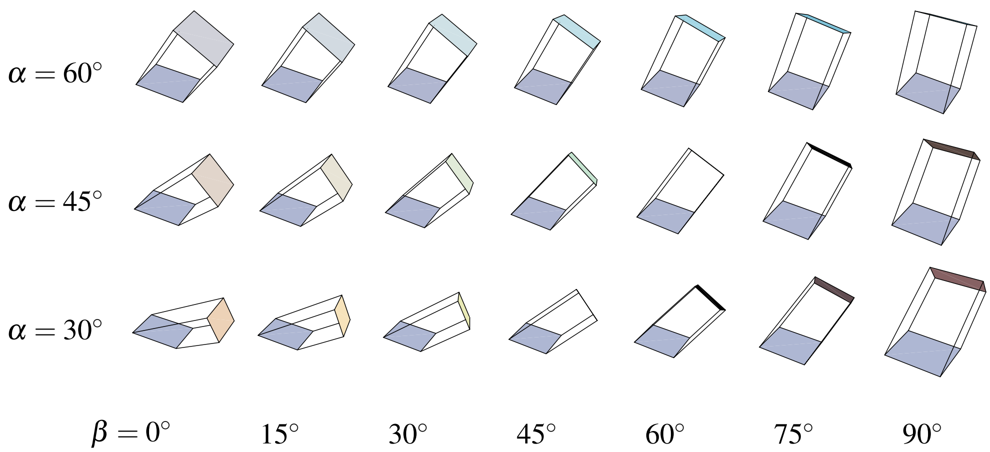

Figure 7: *Beams with same* $1 : \sqrt{2}$ *rectangular cut face, being approaching at various angles*

the cut face from the other side. There can be more directions at which a beam *with the same cross section* can approach the given cut face F. The more symmetries F has, the more possibilities there are for joining the beams at an angle in a matched way.

Figure 8 shows a $1 : \sqrt{2}$ rectangular cut face approached by eight beams (four from each side), all having the same cross section. The joint between beams labeled 1 and 5, and in general between i and $i+4$, is a regular miter joint. The joint between beams labeled 1 and 4 is a regular fold joint. New are the miter joints between, for instance, beams 1 and 6, and between beams 1 and 7.

Note that in case of the regular miter joint, the next beam in the chain lies in the plane perpendicular to the cut face, containing beam 1. In other words, the plane spanned by the beams at the regular miter joint is perpendicular to the cut face. With the new miter joints, the next beam turns sideways at an angle, outside the perpendicular plane containing beam 1. Therefore, we call these **skew miter joints**. The cut face does *not* lie in the interior bisector plane of the skew joint angle, as opposed to the regular miter joint. The cut face does contain the interior bisector line of the joint angle, but the plane spanned by the beams at the joint is *not* perpendicular to the cut face.

New also are the **skew fold joints** between beams 1 and 2, and between beams 1 and 3, where the cut face (= fold face) does not lie in the exterior bisector plane of the joint angle, as opposed to the regular fold joint. The cut face does contain the exterior bisector line of the joint angle. For completeness' sake, we mention that the joint between beams labeled 1 and 8 is a straight extension.

Note that for the four skew joints in Figure 8, the beam has to be turned around, exchanging forward and backward along its axis. That is, if the beam is not forward-backward symmetric, for instance because there are longitudinal arrows on it, then the arrows will change direction at the skew joint. This is the case here because the cut face's reflection symmetries are involved in the skew joints.

A complete characterization of the angles at which beams with the same cross section can form a matched miter or fold joint is given by the following theorem.

> **Miter Joint Angle Characterization Theorem** Let beam B have cross section S and cut face F. The set of all beams with cross section S and cut face F that form a matched joint with B at F consists exactly of those beams obtained from B and its extension through F by applying a symmetry of F. (These symmetries are in 3D, including reflection in the plane containing F).

The beams generated by the symmetries belong in the set, because the symmetries of F transform the cross section into a congruent shape, forming a matched joint at F. That the set contains no other beams is less

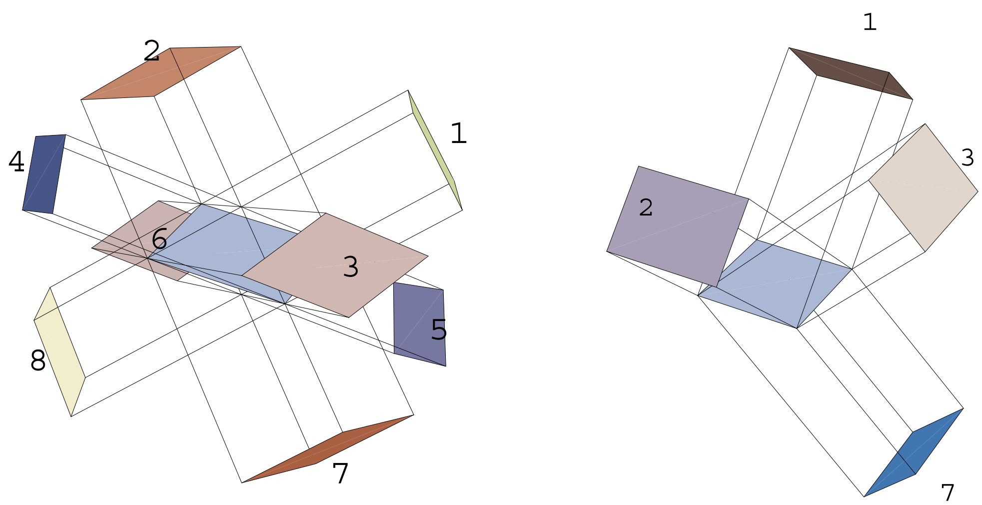

Figure 8: *Beams with same cross section and common* $1:\sqrt{2}$ *rectangular cut face;* $\alpha = 30°$ *and* $\beta \approx 35.26°$ *(left); Beams with rectangular* $1:\sqrt{2}$ *cross section and square cut face;* $\alpha = 45°$ *and* $\beta = 90°$ *(right)*

obvious, and can be proven using linear algebra. That α is the same for all these beams when F is not a line segment, follows from the fact that the *area* of the cross section equals $\sin\alpha$ times the area of F. The theorem can be used to determine at what angles beams with the same cross section can form matched joints.

The theorem shows that there can be 16 different directions at which beams with the *same* cross section approach a square cut face. Eight approach from one side, another eight from the opposite side. This results in one regular miter joint, one regular fold joint, six skewed miter joints, and six more skewed fold joints.

3.3 Joint Angles

Once the possibilities are known for joining beams of a given cross section, it becomes interesting to investigate the angles of the resulting miter and fold joints. For instance, Figure 8 shows all beams with a $1:\sqrt{2}$ rectangular cut face and a particular cross section (a parallelogram), such that the angles are nice: 60° (skew fold joint between beams 1 and 3; regular miter joint between beams 1 and 5), 90° (skew fold joint between beams 1 and 2; skew miter joint between beams 1 and 7), and 120° (regular fold joint between beams 1 and 4; skew miter joint between beams 1 and 6). In this case, we need $\alpha = 30°$ and $\beta = \arctan\frac{1}{\sqrt{2}} \approx 35.26°$, together with those β obtained by the symmetries of the rectangular cut face.

Figure 8 illustrates the three possible miter joints using a $1:\sqrt{2}$ rectangular beam with a square cut face. The regular miter joint between beams 7 and 3 creates an angle of 90°. The skew miter joints between beams 7 and 1, and between beams 7 and 2 both create the —surprising— angle of 120°. With these skew miter joints, the next beam turns sideways at a *right* angle, either to one side, or the other. Therefore, we also call them **transverse miter joints**. Note that because of the symmetries of the square, these transverse joints can be based on rotations rather than reflections. Hence, the beam need not be forward-backward symmetric and can still maintain its directionality across the skew joints.

In general, the relationship between α, β, the symmetries of the cut face, and the joint angles is a rather unpleasant trigonometric expression. Except for special cases, we have only been able to explore it numerically.

4 Artwork

Both objects in Figure 9 trace out a closed path using 24 identical pieces. Each piece has a $1 : \sqrt{2}$ rectangular cross section and a square cut face with $\alpha = 45°$ and $\beta = 0°$ (also see Figure 8, right). The cuts at the two ends are perpendicular (not parallel) to each other, resulting in a trapezoidal side view. The joint angles between such beams are either 90° (regular miter joint) or 120° (transverse miter joint). With four of these pieces one can construct a square picture frame, and with six a regular hexagon. A computer search has determined that there are a total of 62 688 closed paths with 24 of these pieces. The *MathMaker* building box [2] contains these pieces with little magnets at the cut faces to allow easy experimentation.

Figure 9: *Hamilton path on truncated octahedron (left); another closed path with the same pieces (right)*

The object in Figure 9 (left) is a *Hamilton path on a truncated octahedron*. It is highly symmetric, containing 120° transverse miter joints only. The object in Figure 9 (right) has 14 right angles. It is the *right-angle champion* among the 62 688 closed paths. Both objects have 0° torsion.

Figure 10 (left) shows a *Hamilton path on a cuboctahedron*, with miter joints having $1 : \sqrt{2}$ rectangular cut faces. The regular miter joint has an angle of 60°, and the skew miter joints result in 90° and 120° angles (also see Figure 8, left). The beam cross section is a parallelogram with unequal sides. All pieces are identical, with both ends beveled at the same angle $\alpha = 30°$ without rotation. The latter implies that using regular miter joints only results in a planar structure (an equilateral triangle, to be precise). The spatial shape is obtained through the skew miter joints. However, by alternating skew miter joints, these can also yield planar structures, such as a square or a hexagon.

Finally, we illustrate strip folding in Figure 10 (right). This is a trefoil knot exhibiting six-fold (no pun intended) symmetry, having a Möbius twist, that is, it is one-sided. One can obtain it by folding a ‘standard’ Möbius strip and then cutting out the center one-third of that strip.

5 Conclusion

We have explored the miter joint and its variations. The following constraints were assumed:

- a *polyline* in space, that is, a chain of line segments, whose vertices we can control, and whose line segments we wish to thicken;
- the polyline can be a closed chain, that is, it can be a (possibly nonplanar) *polygon*;
- a *polygon* as cross section for the beams used to thicken the line segments;

Figure 10: *Hamilton path on cuboctahedron (left); folded strip figure, named 'Lambiek' (right)*

- all beams have the *same cross section*, and their longitudinal edges are *parallel* to the center line; the beams are *forward-backward symmetric*, i.e., the cross section can be reflected;
- the cross section can itself be a line segment; in that case, we obtain a *strip* (infinitely thin beam);
- edges of adjacent beams (strips) should *match* at the joints;
- a special class is obtained by requiring that all joints have the *same cut face*.

Especially the 3D miter joint variations provide interesting possibilities for constructing attractive structures. For instance, the cut face need not lie in the bisector plane of the (interior) angle. In that case, we get a skew miter joint. These offer additional angles, while using the same cross section for all beams.

'Folding' is another way of joining beams. In practice, this is especially interesting for strips. The fold line for a strip lies in the bisector plane of the exterior angle. 'Folding' a non-degenerate beam gives rise to self-intersection. The fold face need not lie in the exterior bisector plane, even when the cross section is preserved, giving rise to a skew fold joint.

The mathematics is not particularly difficult: (oblique) parallel projection, possibly repeated, and symmetry groups. We have given a complete characterization of the relationship between cross section, cut face, and possible joint angles.

The application in practice is fraught with complications. In particular the inverse problems often do not have an elementary analytic solution. In that case, one needs to resort to numerical approximations. Therefore, it is useful to explore special classes of curves, for example, whose vertices are lattice points.

It is tempting to relax some of the constraints. Polylinks, different cross sections across joints, continuous spatial curves, and higher dimensions offer ample opportunities for further investigation.

Acknowledgments

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