Neo-Riemannian Geometry

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Abstract

This paper considers groups of musical "contextual" transformations, the most famous of which is a group of bijections between minor and major triads described by the music theorist Hugo Riemann (1880). Mathematically, contextual transformations act on chords or melodies and commute with transposition (shifting by the same number of pitches in the same direction). This is important because most people naturally identify two melodies or chord progressions as "the same" if one is a transposition of the other. Music theorists have studied contextual transposition and inversion groups extensively; in particular, Lewin (1987), Kochavi (1998), and Fiore and Satyendra (2005) used discrete group theory, while Clough (1998) and Gollin (1998) considered symmetries of discrete geometric spaces. The action of contextual transpositions and inversions on the *continuous* geometric "voice-leading spaces" of Callender, Quinn, and Tymoczko (2008) reveals subtleties that do not arise in the traditional discrete approach. I propose two ways of understanding contextual transpositions and inversions, one employing a "bundle" construction and the other representing contextual transformations as a family of linear transformations. The first involves topological group theory, the second dynamics. I discuss the advantages and drawbacks of each method.

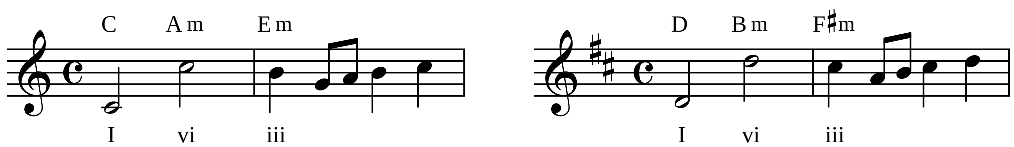

Figure 1: *"Somewhere over the Rainbow" in the keys of C major and D major.*

1. Introduction

Consider two versions of "Somewhere Over the Rainbow." The first version begins with a C note and a C major chord. (The letter names above the staff indicate chords that might be played on a guitar or piano to accompany the singer.) The second version has been *transposed*—shifted up in pitch[1]—so that it starts on a D note with a D major chord. Our brains naturally identify these two realizations of the melody as "the same." In fact, unless the two versions are played in succession, most people—including many highly trained musicians—cannot distinguish between them.

Because many chordal relationships are invariant under transposition, Western musicians have traditionally labeled chords with Roman numerals indicating their relationship to a tonal center. Each numeral represents the scale degree of a chord's root note, with a capital letter for a major triad and a lowercase letter for a minor triad. For example, the C major chord is labeled *I* in the key of C major. Although one rarely finds two different musical pieces with identical melodies, chord progressions that are the same up to transposition are common. For example, "I Got Rhythm" and "Blue Moon" start with the progression *I*-*vi*-*ii*-*V*-*I*. Figure 2 shows a voicing of this progression in two different keys (C major and D major).

The examples considered above share another important feature: they involve only two types of chords, major and minor triads. These are closely related. A major triad is a minor third stacked on top of a major third, while a minor triad is a major third stacked on top of a minor third. Music theorists say that major and minor triads constitute a "set class."

[1]*Pitch* is frequency measured on a logarithmic scale. A piano keyboard is a linear representation of pitch; the smallest interval between two keys on the piano is called a *semitone* and equals one unit of pitch.

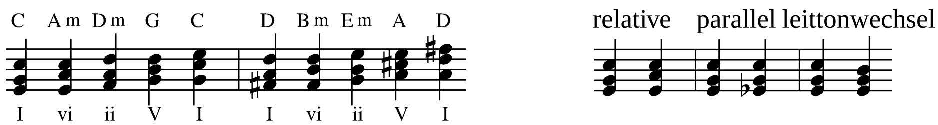

Figure 2: *The* I-vi-ii-V-I *progression; the relative, parallel, and leittonwechsel progressions.*

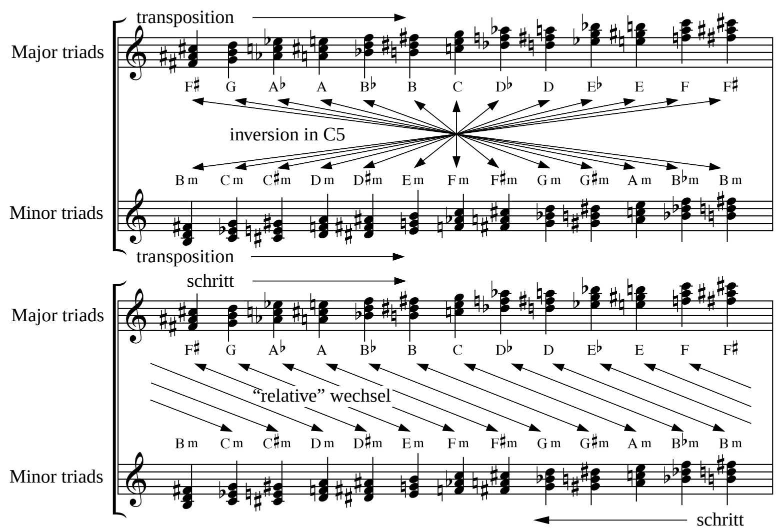

Figure 3: *Transposition, inversion, schritt, and the “relative” wechsel acting on major and minor triads in equal temperament.*

There are several relevant bijections between major and minor triads. Inversions, depicted in Figure 3 (top), reflect every chord around some fixed “origin” (here, C5[2]); thus, the inversion of a major triad is a minor triad and vice versa. Inversion does not commute with transposition. Most people would not identify the inversion C major–F minor as “the same progression” as the inversion D major–E♭ minor.

In contrast, the distance between (C4, E4, G4) and its inversion in G♯4 (A4, C5, E5)—also known as its relative minor—is the same as the distance between (D5, F♯5, A5) and its inversion in A♯5, or relative minor. Contextual inversions, called *wechsels*, capture this similarity: the axis of inversion “follows” a chord as it is transposed. The composition of two wechsels is a contextual transposition, or *schritt*, that moves a chord and its inverted form by the same amount in opposite directions. Mathematically, schritts and wechsels commute with transposition and inversion (though not with each other).

The *neo-Riemannian transformations*, first described by the music theorists Arthur von Oettingen [10] and Hugo Riemann [11], form a group of bijections between the set of major triads and the set of minor triads that commute with transposition and inversion. The wechsels “relative,” “parallel,” and “leittonwechsel” generate this group (Figure 2, right). Such progressions are ubiquitous, from early music to rock. The *I-vi-ii-V* progression contains two neo-Riemannian progressions ($I \leftrightarrow vi$ and $ii \leftrightarrow V$).

Neo-Riemannian transformations belong to a larger class of so-called *contextual transformations*—“contextual” because their action depends on the objects they act on. Music theorists, most notably Lewin [8], have extended the definition of contextual transformations to apply to other chords and even ordered sequences of pitches. They have not restricted their investigations to chords common in tonal music; contextual transformations appear in both tonal and atonal music.

[2]The “5” indicates the octave: C4 is middle C and C5 is an octave above middle C.

2. Geometrical Music Theory

This paper interprets contextual transpositions and inversions as actions on the geometric spaces described by Callender, Quinn, and Tymoczko [12, 1] (henceforth, CQT). Although schritts and wechsels are staples of traditional musical set theory, they have not yet been incorporated into geometrical music theory. In the following pages, we consider how schritts and wechsels can be realized as group actions on CQT spaces.

Formally, pitch is frequency measured on a logarithmic scale. There are twelve units of pitch (semitones) to an octave. Pitches lie on a continuum; integer pitches form twelve-tone equal temperament (12-tet). An ordered multiset of pitches corresponds to a point in n-dimensional space ($\mathbb{R}^n$).

Musicians commonly recognize several groups acting on ordered pitch space: the permutation group ($\mathbf{P}$), whose elements reorder multisets of pitches; the group of octave shifts ($\mathbf{O}$), which move an individual pitch in a multiset up or down by some number of octaves; the transposition group ($\mathbf{T}$), consisting of translations shifting all the pitches in a multiset up or down by a fixed amount; and a two-element inversion group ($\mathbf{I}$), or reflection in the origin.

These groups induce equivalence relations on $\mathbb{R}^n$ by identifying points in the same orbit. Some equivalence classes have familiar names. The equivalence class of a single pitch under octave shift is called its *pitch class*. Pitch classes are indicated by letter names ("C"). A *chord* is an unordered multiset of pitch classes. For example, any combination of C, E, and G pitch classes forms a "C major chord." The set of major chords is an example of a *chord type* (also called a *transpositional set class*)—a set of chords related by transposition. Finally, a *set class* is an equivalence class of chords related by transposition and inversion. "Minor and major triads" form a set class.

The groups $\mathbf{O}$, $\mathbf{P}$, $\mathbf{T}$, and $\mathbf{I}$ are all generated by affine linear transformations on $\mathbb{R}^n$ (for example, addition of $(12, 0, \ldots, 0)$ is an element of $\mathbf{O}$). Identifying ("gluing together") orbits of these groups produces a family of singular quotient spaces called *orbifolds*. For example, pitch classes inhabit the circle $\mathbb{R}/\mathbf{O} = \mathbb{R}/12\mathbb{Z}$; ordered multisets of pitch classes inhabit the torus $\mathbb{R}^n/\mathbf{O} = \mathbb{T}^n$. Chords lie in $\mathbb{R}^n/\mathbf{OP}$ and chord types in $\mathbb{R}^n/\mathbf{OPT}$. Set classes are orbits under $\mathbf{OPTI}$ equivalence. Cardinality equivalence ($\mathbf{C}$), which identifies points whose coordinates form the same set, ignoring multiplicities, is an additional equivalence relation, though not one arising from a group action. *CQT spaces* are quotients of n-dimensional ordered pitch space by any combination of the five $\mathbf{OPTIC}$ equivalence relations. (See Table S2 in [1] for a full description of CQT spaces).

A directed line segment in n-dimensional ordered pitch space corresponds to a mapping from one musical object to another. If we think of each coordinate as representing an instrumental "voice," a directed line segment indicates how voices move in time; any musical score can be represented as a succession of directed line segments in $\mathbb{R}^n$ or any of its quotient spaces. Musicians call these mappings *voice leadings*. Voice leadings may inherit "length" from some distance function in ordered pitch space (see [5]).

3. A Bundle Representation

A chord $\mathbf{v}$ is a multiset of points on the pitch class circle. Let T_x represent transposition (rotation) up by x pitches and I represent inversion (reflection) in the axis passing through the origin (any other inversion is a composition of I and a transposition). Transposition and inversion anticommute ($T_x I = IT_{-x}$). The $\mathbf{TI}$ group is a topological group isomorphic to the orthogonal group $O(2)$, the isometries of $\mathbb{R}^2$ that fix the origin. Topologically, $O(2)$ is two disconnected circles; $SO(2)$ is the component containing the identity. (In the discrete group theory normally used by music theorists, the group of transpositions and inversions is isomorphic to the dihedral group D_{24}.)

Traditional music theory (see Forte [4]) classifies chords based on their memberships in set classes—equivalence classes of chords related by transposition and inversion. Each set class is an orbit under the action of the $\mathbf{TI}$ group. In most cases, the $\mathbf{TI}$ group acts transitively on set classes, meaning that the identity is the only group element that fixes elements of the set class. Exceptions occur at "singular" chords possessing additional rotational or inversional symmetries. For example, the singular chord $\{2, 8\}$ is fixed by T_6, T_4I, and $T_{-2}I$.

The orbit of any chord without inversional symmetry is topologically identical to the $\mathbf{TI}$ group. That is, it consists of two disconnected circles, arbitrarily labeled V^+ and V^-; every point on one of the circles represents some member of the set class. Figure 4, left, depicts the action of transposition and inversion on this orbit. (Note that this is not a picture of the pitch class circle. Points, rather than multisets of points, represent chords.) The orbit of a chord with inversional symmetry consists of only one circle. If a chord has nontrivial rotational but not inversional symmetry, its set class is two circles, but "smaller" in that they are fixed by transposition of $12/k$ pitches for some integer $k > 1$.

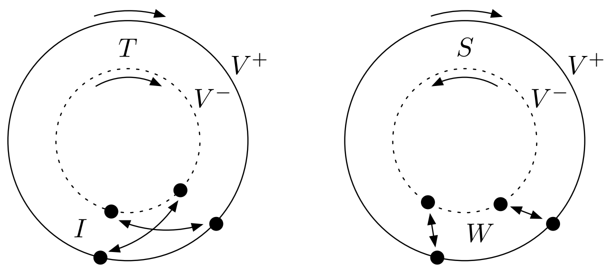

Figure 4: *T, I, S, and W acting on a nonsingular set class.*

We define a schritt S_x to be rotation of V^+ up by x pitches and V^- down by x pitches and a wechsel W to be a bijection between the two circles that commutes with transposition. Figure 4, right, depicts the action of schritts and wechsels. Like transpositions and inversions, schritts and wechsels anticommute—that is, $S_xW = WS_{-x}$. If we think of elements of the $\mathbf{TI}$ group as acting on the left, then S_x and W are precisely the *right* actions of transposition and inversion (physicists call this construction a torsor—see Lavelle [7]). We refer to this group as the $\mathbf{SW}$ group.

Hook [6] showed that the $\mathbf{SW}$ group acting on triads is the dual to the $\mathbf{TI}$ group in the sense that each group is the centralizer of the other; Fiore and Satyendra [2] extended this result to ordered pitch class sets in k-tet. They described a schritt/wechsel group acting on the rational points in $\mathbb{T}^n$ as the *generalized contextual group*.

With the exception of transposition, defining these transformations require some (perhaps arbitrary) choices. Schritts require the designation of one circle as "positive" and the other as "negative," inversions require the choice of a preferred axis, and wechsels require an alignment of V^+ and V^-. It suffices to define one inversion and one wechsel for every orbit, since all the others are compositions of inversion and transposition or wechsel and schritt. There exists a "reference point" $\mathbf{b}$ for which $W(\mathbf{b}) = I(\mathbf{b})$; this base point determines the action of W and I on the entire orbit. The reference point is unique up to equivalence modulo 6.

Our goal is to define schritts and wechsels on continuous set class space. We first consider the product space $\mathbb{R}^n/\mathbf{OPTI} \times O(2)$. This space has two connected components, each isomorphic to $\mathbb{R}^n/\mathbf{OPTI} \times SO(2)$, and natural left and right $O(2)$ actions $h(\sigma, g) = (\sigma, hg)$ and $(\sigma, g)h = (\sigma, gh)$, respectively, for σ a set class and g, h in $O(2)$. The product space is a *principal homogeneous space* for the action of $O(2)$; each copy of $O(2)$ is a *fiber*.

Transposition is clearly continuous. The challenge is to make the choices required to define schritts, wechsels, and inversions as continuous maps on the product space. A *section* is a continuous map from set class space $\mathbb{R}^n/\mathbf{OPTI}$ to $\mathbb{R}^n/\mathbf{OPTI} \times O(2)$ such that each set class σ is mapped to some pair (σ, f), where f is an element of the fiber associated to σ. The positive component—the one each V^+ inhabits—contains the image of $\mathbb{R}^n/\mathbf{OPTI}$. The canonical I and W are inherited from $O(2)$.

Because of the existence of singular set classes, the $\mathbf{TI}$ and $\mathbf{SW}$ groups do not act simply transitively on any product space. In order to model the transposition/inversion group, we form a singular bundle B, with the fiber at each point representing a group of transpositions and inversions that act transitively on that set class. The fiber associated to an inversionally symmetric set class has only one component; for set classes dividing the octave evenly into p parts, the fiber is one or two copies of $\mathbb{R}/(12/p)\mathbb{Z}$ rather than $\mathbb{R}/12\mathbb{Z}$. This fiber bundle is similar in structure to $\mathbb{R}^n/\mathbf{OP}$.

Transpositions act continuously, even on a singular bundle. However, a schritt S_x should move points on V^+ and V^- by x pitches in opposite directions. One cannot define a schritt S_x that acts continuously on B, as there exist inversionally symmetric set classes arbitrarily close to pairs of circles V^+ and V^- along which S_x acts in opposite directions. However, there are *regions* of set class space within which one can define continuous schritts.

Fiore and Satyendra [2] described a structure on which the $\mathbf{TI}$ and $\mathbf{SW}$ groups act simply transitively by essentially allowing every set class—even those with inversional symmetry—to have *two* canonical representatives, one designated an inversion. Their construction may be realized by a section of the product space rather than a section of the singular bundle B.

4. Linear Transformations on Ordered Pitch Class Space

Transposition and inversion are affine linear transformations acting on ordered pitch class space $\mathbb{T}^n$ or ordered pitch space $\mathbb{R}^n$. Is there a natural representation of contextual transpositions and inversions as affine linear transformations?

Following Lewin [8], we refer to an ordered multiset of pitches or pitch classes as a *pitch segment* or *pitch class segment*. Suppose the affine linear transformation on n-dimensional pitch class space $\mathbf{z} \mapsto M\mathbf{z} + \mathbf{b}$ commutes with transposition, where M is a $n \times n$ matrix and $\mathbf{b}$ is in $\mathbb{R}^n$. Let $\mathbf{1} = (1, 1, \ldots, 1)$. Then for $\mathbf{z} \in \mathbb{R}^n$ and c a real number,

$$M(\mathbf{z} + c\mathbf{1}) + \mathbf{b} = M\mathbf{z} + \mathbf{b} + c\mathbf{1},$$

so M fixes $\mathbf{1}$. Since $\mathbf{b}$ does not depend on $\mathbf{z}$ ($\mathbf{b}$ determines a voice leading), we restrict our attentions to linear transformations $\mathbf{z} \mapsto M\mathbf{z}$. All transformations of this form also commute with inversion. It seems reasonable that no nonzero subspace of $\mathbb{T}^n$ be mapped to zero, so we assume that M is invertible. Moreover, M should preserve octave equivalence; that is,

$$M(\mathbf{z} + (0, \ldots, 0, 12, 0, \ldots, 0)) \equiv \mathbf{z} \pmod{12},$$

which implies that the entries of M are integers and hence the determinant of M is ± 1. Therefore, M is an element of the general linear group $GL(n, \mathbb{Z})$. (Transformations that do not preserve octave equivalence are rarely useful in music theory.)

Let $\mathcal{G}$ be the group of invertible linear transformations on the torus $\mathbb{T}^n$ that commute with transposition and inversion: $\mathcal{G} = \{M \in GL(n, \mathbb{Z}) : M\mathbf{1} = \mathbf{1}\}$. We will investigate the structure of $\mathcal{G}$. In particular, we seek a workable definition of the subgroups of $\mathcal{G}$ consisting of transformations that preserve chord type or set class—these will be candidates for the contextual transposition group and the contextual transposition/inversion group, respectively.

At this point, it is convenient to choose a basis for $\mathbb{T}^n$ so that $\mathbf{1}$ is a basis vector. Using the basis $\{\mathbf{1}, (-1, 1, 0, \ldots, 0), (-1, 0, 1, 0, \ldots, 0), \ldots, (-1, 0, \ldots, 0, 1)\}$, every point $(z_0, z_1, \ldots, z_{n-1})$ can be written in the form $(x_0 \,|\, x_1, x_2, \ldots, x_{n-1})$ where $x_0 = z_0$, $x_1 = z_1 - z_0$, $x_2 = z_2 - z_0$, and so on. (Think of x_0 as the first note in the segment, and the other x_i's as intervals measured from the first note; "Twinkle, Twinkle" is represented $(0 \,|\, 0, 7, 7, 9, 9, 7)$ in C major and $(2 \,|\, 0, 7, 7, 9, 9, 7)$ in D major.) With this basis, every element of $\mathcal{G}$ can be written uniquely in the form

$$\left(\begin{array}{c|ccc} 1 & a_1 & \cdots & a_{n-1} \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right)$$

where $(a_1, \ldots, a_{n-1}) = \mathbf{a}$ is in $\mathbb{Z}^{n-1}$ and A is in $GL(n-1, \mathbb{Z})$. We represent elements compactly as pairs $\langle \mathbf{a}, A \rangle$. Multiplication in $\mathcal{G}$ follows the rule

$$\langle \mathbf{a}, A \rangle * \langle \mathbf{b}, B \rangle = \langle \mathbf{a}B + \mathbf{b}, AB \rangle.$$

The set $\{\langle \mathbf{a}, id \rangle : \mathbf{a} \in \mathbb{Z}^{n-1}\}$, where id represents the $(n-1) \times (n-1)$ identity matrix, is a normal subgroup of $\mathcal{G}$ that is isomorphic to $\mathbb{Z}^{n-1}$. The following sequence is exact:

$$0 \longrightarrow \mathbb{Z}^{n-1} \longrightarrow \mathcal{G} \longrightarrow GL(n-1, \mathbb{Z}) \longrightarrow 0.$$

We see that $\mathcal{G}$ is isomorphic to the semidirect product $\mathbb{Z}^{n-1} \rtimes GL(n-1, \mathbb{Z})$. Let CT represent the normal subgroup $\{\langle \mathbf{a}, id \rangle : \mathbf{a} \in \mathbb{Z}^{n-1}\}$.

Both CT and $GL(n, \mathbb{Z})$ appear in musical contexts. Contextual transpositions are transformations that transpose a pitch class segment and its inversion in opposite directions by equal amounts. Thus—if one agrees with the assumptions made in this section of the paper—CT is a contextual transposition group. In tuning theory, two tunings defined by n generating intervals are considered identical if the intervals of one can be mapped to the intervals of the other (modulo octave equivalence) by a linear transformation in $GL(n, \mathbb{Z})$ [9].

We would also like to find a subgroup of $\mathcal{G}$ that is a plausible candidate for the group of contextual transpositions and inversions. The element $\langle \mathbf{a}, -id \rangle$ generates a group of order two; it represents the wechsel that inverts the intervals in a pitch segment while fixing its first note.[3] We form the exact sequence

$$0 \longrightarrow \mathbb{Z}^{n-1} \rtimes \mathbb{Z}_2 \longrightarrow \mathcal{G} \longrightarrow PGL(n-1, \mathbb{Z}) \longrightarrow 0.$$

(PGL is the projective linear group $GL(n-1, \mathbb{Z})/\{\pm 1\}$.) Let CI denote $\mathbb{Z}^{n-1} \rtimes \mathbb{Z}_2 = \{\langle \mathbf{a}, \pm id \rangle\} \subset \mathcal{G}$.

The group CI of contextual inversions and transpositions is normal in $\mathcal{G}$ and contains CT as a normal subgroup; $\mathcal{G}$ is the semidirect product of CI and $PGL(n-1, \mathbb{Z})$. Elements of CI act on points $(x_0 \,|\, \mathbf{x}) := (x_0 \,|\, x_1, \ldots, x_{n-1})$ in the pitch class torus by

$$\langle \mathbf{a}, \pm id \rangle (x_0 \,|\, \mathbf{x}) = (x_0 + \mathbf{a} \cdot \mathbf{x} | \pm \mathbf{x}).$$

(In other words, the amount by which the segment or its inversion is transposed is a linear combination of its intervals.)

Although CI acts simply transitively on the torus as a whole, CI does not act simply transitively on all *points* in $\mathbb{T}^n$—in fact, every pitch class segment with rational intervals is fixed by some nontrivial subgroup of CI. For example, all 12-tet segments are fixed by $\langle 12\mathbf{a}, id \rangle$. Taking the quotient of CI by $12\mathbb{Z}^{n-1} \rtimes \mathbb{Z}_2$ produces the "12-tet contextual transposition/inversion group" $\mathbb{Z}_{12}^{n-1} \rtimes \mathbb{Z}_2$.[4]

When we take voice leadings into account, the CT and CI groups are anything but well behaved. Not only are they not isometries of $\mathbb{T}^n$, but they are actually "chaotic": For any pitch class segment $\mathbf{a}$ and

[3] Another subgroup of $\mathcal{G}$ that produces a similar structure is generated by *retrograde*, or reversing the order of a pitch class segment. In some situations it will be more convenient to use this group, or perhaps to use both retrograde and contextual inversion.

[4] Hook [6] studied an isomorphic group for $n = 3$ extensively, in the context of transformations of pairs of 12-tet major and minor triads (*uniform triadic transformations*). There exists an analogous group in any dimension that acts transitively on $(n-1)$-element linearly independent sets of pitch class segments.

any nontrivial contextual transposition A, there exists a pitch class segment $\mathbf{b}$ arbitrarily close to $\mathbf{a}$ that is mapped arbitrarily far from $\mathbf{a}$ by repeated application of A to both segments. Voice leadings (line segments) are badly distorted by these groups. This is rather discouraging, as one of the most attractive features of CQT spaces is that they possess a natural notion of "size." Tymoczko [13] observes that music theorists sometimes seem to assume that there is a notion of size *inherent* in transformation groups—however, in many reasonable musical situations it is essential to make this notion explicit.

5. Examples

David Lewin was particularly interested in contextual transpositions and inversions that preserve at least one common tone. Examples from his work include the so-called RICH, TCH = RICH^2, MUCH, TLAST, TFIRST, FLIPEND, and FLIPSTART transformations [8]. These are all elements of CI. Is CI generated by common-tone preserving transpositions and inversions? The answer is yes. A set of generators is

$$\langle(0,\dots,0),-id\rangle, \langle(1,0,\dots,0),id\rangle, \langle(0,1,0,\dots,0),id\rangle, \langle(0,\dots,0,1),id\rangle$$

(that is, the contextual inversion that fixes the first element of a pitch class segment, plus a linearly independent set of contextual transpositions that preserve a common tone).

5.1. The PLR group. The CI group acting on $\mathbb{T}^3$ is generated by the linear transformations $W_{12} = \langle(1,0),-id\rangle$, $W_{13} = \langle(0,1),-id\rangle$, and $W_{23} = \langle(1,1),-id\rangle$ (note that W_{ij} exchanges the ith and jth elements of the pitch class segment). It has the presentation $\langle W_{12}, W_{13}, W_{23} : W_{12}^2 = W_{13}^2 = W_{23}^2 = (W_{12}W_{13}W_{23})^2 = 1\rangle$, its elements can be represented by paths in a honeycomb lattice (its Cayley graph), and it is isomorphic to $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$.

This departs from musical tradition in several ways. The CI group does not act simply transitively on the orbit of any pitch segment that can be embedded in equal temperament. Moreover, the size of an orbit depends on the pitch class segment chosen: if a pitch class segment can be embedded in k-tone equal temperament, then its orbit under the action of CI can have no more than $2k$ elements. So, for example, the orbit of CI acting on $(1,5,9)$ can have no more than six elements, since $(1,5,9)$ can be embedded in 3-tone equal temperament. In contrast, orbits of pitch class segments containing irrationally-related intervals (*e.g.* the just-tempered major triad $(0, 12\log_2(5/4), 12\log_2(3/2))$) are infinite; in this case, CI acts simply transitively. Although an infinite orbit is true to Oettingen and Riemann's original conception of these transformations, the usefulness of this group in the irrational case is considered and rejected by Lewin [8, 8.2.1].

The PLR group is the restriction of the CI group to the orbit of a equally tempered major triad. Let M be the orbit of $(0,4,7)$ under the CI group:

$$M = \{\pm(x,4,7) : x \in \mathbb{Z}_{12}\}.$$

The stabilizer of $(0,4,7)$ is the cyclic subgroup of order 12 $\{\langle(a,b),id\rangle : 4a+7b \equiv 0 \pmod{12}\}$ generated by $\langle(5,4),id\rangle$. The quotient of CI by this group is the PLR group, which is isomorphic to $\mathbb{Z}_{12} \rtimes \mathbb{Z}_2$. The generators of CI are the familiar neo-Riemannian transformations: W_{12}, W_{13}, and W_{23} act as R, P, and L respectively.[5]

5.2. Generated sequences. Let $A = \langle(1),id\rangle$ on $\mathbb{T}^2$. Successive application of A to an interval $(0,g)$ gives the sequence of "fifths" $(0,g),(g,2g),(2g,3g),\dots$. The map $g \mapsto ng \pmod{12}$ is the chaotic dynamical system depicted in Figure 5. One result of this fact is that you can't simultaneously tune fifths and octaves—a fact that has plagued musicians for centuries! The orbit of the equally tempered major third has three points and the orbit of the equally tempered perfect fifth has twelve points; the orbit of any irrational interval is dense in the octave.

[5] This construction also appears in [3]. Note that the order of a pitch segment is crucial. For example, $W_{12}(7,0,4) = (0,7,3)$, which is not the relative minor of $(0,4,7)$.

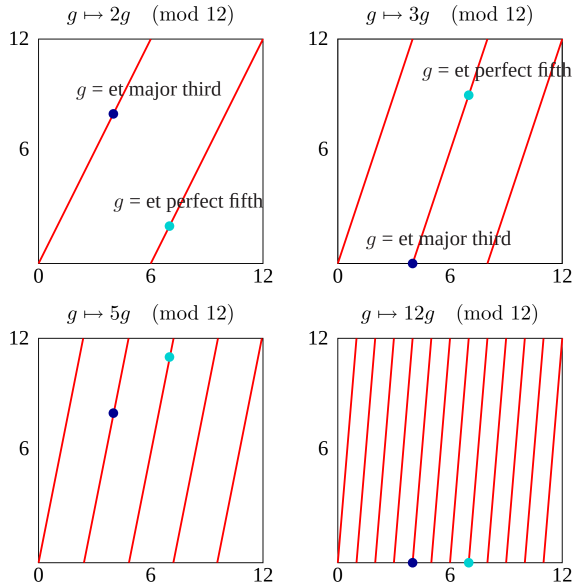

Figure 5: The chaotic dynamical system $g \mapsto ng \pmod{12}$

6. Conclusion

This paper explores two mathematical models for contextual transposition and inversion groups acting on CQT spaces. The process of adapting algebraic structures to continuous spaces points out differences that are concealed by discrete set theory. My feeling is that the first is more consistent with the literature, as contextual transpositions and inversions are generally described as actions on set classes. However, the second representation does arise in music theory, both in the work of Lewin and in the context of tuning and scale theory.

It is not clear whether either of the models discussed is the "right" way to think about contextual transformations in CQT spaces. Music, not mathematics, should motivate that investigation.

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