A Computer Aided Geometric Model of a Ten-Plane Polyhedral Transformation

*Robert McDermott\(^1\) and Will Hawkins\(^2\)
Center for High Performance Computing\(^1\) and School of Computing\(^2\)
University of Utah
Salt Lake City, Utah 84112, USA
Email: robert.mcdermott@utah.edu

Abstract

A computer aided geometric model for a polyhedral transformation was written in C++, OpenGL and GLUT. This geometric model allows a user to interactively change the shape of a closed convex polyhedron derived from 10 face planes in a fundamental region. Given different inputs, a different shaped polyhedron was output. For some inputs, the output was a Platonic or Archimedean polyhedron. For other inputs a more complex polyhedron was output. This transformation is based on a fundamental region that is formed by 3 symmetry planes intersecting at the centroid of a polyhedron. This fundamental region is combined with 3 different sets of transformations to produce a polyhedron for either a tetrahedral, an octahedral, or an icosahedral family of polyhedra. Using the interactive software, a polyhedron transforms smoothly and continuously from a regular polyhedron, to a truncate polyhedron, to a dual polyhedron, to a stellate polyhedron along with many transitional polyhedra.

1. Introduction

This paper extends a previous paper presented on Coxeter Day at Bridges 2005 \([1]\) where a 3 plane computer aided geometric model was presented. This 3 plane model was presented using 10 minutes of computer animation. More recently work has produced interactive software to extend that 3 plane computer model to a 10 plane computer model. Software using Microsoft Visual Studio C++ for modeling, OpenGL for graphics, and GLUT for interaction has resulted. A video documentary was produced to show an interactive experience with this software. Subsequently, a DVD was authored to include this documentary \([2]\).

The 3 plane model produces both Platonic and Archimedean polyhedra, as well as, many transitional polyhedra. The 3 plane model uses 9 vectors from 3 foundational polyhedra; a regular polyhedron, its dual and their mid-figure. For example: the tetrahedral family uses 9 vectors, 3 each, from a tetrahedron, a dual tetrahedron and an octahedron. In addition the octahedral family uses vectors from an octahedron, a cube and a cubeoctahedron, where the icosahedral family uses vectors from an icosahedron, a dodecahedron and an icosidodecahedron. A tri-linear combination of the 9 vectors from each of these 3 families produces 3 vectors for a polyhedron that transforms its shape in the computer animation.

The 10 plane model uses 40 vectors from 4 foundational polyhedra. The foundational polyhedra are a regular polyhedron produced by adding 7 planes to the initial 3 plane model, a truncate polyhedron, a dual polyhedron, and a stellate polyhedron. Many polyhedra are formed when transitioning between these 4 foundational polyhedra.
In 1982 Haresh Lalvani, an architecture faculty at Pratt Institute in Brooklyn, New York, exhibited his polyhedral transformations. Many paper models were displayed along with descriptions and drawings. This work appeared in his Ph.D. dissertation completed under Buckminster Fuller at the University of Pennsylvania [3]. His 10 plane transformation was depicted by a series of drawings in his dissertation and physical models in his exhibit.

The initial 3 plane model and the subsequent 10 plane model use C++ code LIB3D written by Pat Hanrahan, who is currently a computer science faculty at Stanford University. This code was first written in C when we were colleagues at NYIT Computer Graphics Laboratory. The inputs for the procedures are vectors, points, lines, and planes. The outputs are also vectors, points, lines, and planes derived from geometric relationships between the inputs. For example, the intersection of 3 planes is a point. The LIB3D code leads the computer aided geometric design model to be a series of geometric relationships, with the final output being a set of points for polygons that are displayed as a polyhedron.

2. Explosion-Implosion Polyhedral Transformation

Haresh named a polyhedral transformation as an Explosion-Implosion. In Figure 1 an Explosion-Implosion is seen as a vertex exploding to a face, then imploding to a vertex. An edge explodes to a rectangle, then to a square. A square implodes to a rectangle, then to an edge, that is orthogonal to the initial edge. A face implodes to a vertex, then explodes to a face. In Figure 1 a vertex of a blue icosahedron (a) explodes to a red pentagon (b,c,d,e,f) of a red dodecahedron (g). An edge of an blue icosahedron (a) explodes to a green rectangle (b,c), then a green square (d) and implodes to a green rectangle (e,f), and then to an edge of a red dodecahedron (g). Finally, a blue icosahedron face (a) implodes to a blue vertex of a red dodecahedron (g). Note that an edge of the blue icosahedron (a) is orthogonal to an edge of the red dodecahedron (g).

![Figure 1: Vertices, edges, and faces are exploding and imploding.](image)

3. A Hypercube with a Unit Edge Length as a Reference Space

The 3 plane model has a cube with unit edges for a reference space for its transformation as there are 3 Explosion-Implosion transformations for a vertex, an edge and a face Figure 2a. The 10 plane model has 4 foundational polyhedra using a hypercube with a unit edge as a reference space for its polyhedral transformation, Figure 2b. A point in a unit edge length reference space has coordinates that vary between 0 and 1 and represents a different polyhedron. A regular polyhedron from these models has edge lengths that vary between 0 and 1. Corner (1,1,1) of the reference cube is the same polyhedron as corner (1,0,0,0) of the reference hypercube. Corner
(1,1,1,1) of the reference hypercube has 10 faces visible in the fundamental region, 1 face for each of the 10 vectors and 10 planes in the fundamental region of the model Figure 2c and Figure 3a.

Figure 2: A reference cube (a), a hypercube (b) 10 normal vectors of fundamental region(c).

4. Fundamental Region for a 10 Plane Model

Three primary faces; red(1), green(2) & blue(3) for a regular polyhedron 3(b) are orthogonal to symmetry vectors, at the corners of the fundamental region. Three secondary faces; yellow(4), cyan(5), and magenta(6) for a stellate 3(e) are orthogonal to vectors at the mid-sides of the fundamental region. A tertiary pink face (7) is orthogonal to a vector between a red & white face and a magenta & yellow face. A tertiary lime face (8) is orthogonal to a vector between a green & white face and a yellow & cyan face. A tertiary aqua face (9) is orthogonal to a vector is between a blue & white face and a cyan & magenta face. In the center is a white face (10) for the 3 truncate faces Figure 3c & a single dual face Figure 3d.

Figure 3: faces (a), regular (b), truncate (c), dual (d), stellate (e) of fundamental region.
5. Polyhedra from the 10 Plane Model with *Hypercube* Reference Space

A *polyhedron* with different *edge* lengths is considered a different polyhedron. *Point* (1,0,0,0) represents a *regular polyhedron* with icosahedronal symmetry (5,2,3), *Figure 4a*. A *truncate polyhedron* is referenced at *point* (0,1,0,0), *Figure 4b*, a *dual polyhedron* is referenced at *point* (0,0,1,0), *Figure 4c*, and a *stellate polyhedron* is referenced at *point* (0,0,0,1), *Figure 4d*. The polyhedra referenced at the other corners of the *hypercube* appear in *Figure 4e*, *Figure 5*, and *Figure 6*. There are 15 non-null *polyhedra* at the corners of the *hypercube* reference space.

Interacting with software to observe the display of polyhedra continuously changing shape, has been very stimulating for those who are interested in polyhedra. Will and I plan a video documentary of an instructive experience as well as author it to a DVD.
6. Implementation for a 10 Plane Explosion-Implosion

6.1. 4 Foundational Polyhedra. This implementation uses 10 normal vectors, Figure 2c, to position 10 planes for each foundational polyhedron. Once 10 planes have been positioned, sets of 3 planes intersect in a point. Sets of these points combine to form face polygons. A set of 10 faces are numbered and displayed in Figure 3a. These point coordinates for the faces are transformed with symmetry transformations to display full closed convex polyhedra in Figures 4, 5, and 6.

6.2. Vectors. Four sets of 10 vectors were computed from a regular, truncate, dual and stellate polyhedron [4]. For each foundational polyhedron, 1 normal vector was computed for each of the 10 faces in the fundamental region, Figure 3b. These normal vectors lie on the boundary or in the interior of the fundamental region Figure 2c. These 40 vectors are blended in 4 quad-linear equations with 4 coordinates from a point within the unit edge hypercube to produce a single set of 10 vectors. These 10 vectors are normalized so that they become meaningful vectors for 10 face planes.

6.3. Planes, Points, Faces. With this set of 10 face planes and the 3 symmetry planes on the sides of the fundamental region, a set of 21 points are computed to display 10 faces. Three planes intersect to compute each point. This implementation works correctly for p, q, and r values of (1,1,1). A solution for the other values of p, q, and r between 0 and 1 is currently under development.

Viewing the colored normal vectors in Figure 2c led to an understanding of the locations of faces for this 10 plane computer model. The fundamental regions in Figure 7a displays implosion-explosion transformations of vertices, edges, and faces by the 10 plane model.

7. Display and Interaction

7.1. Display. In addition to displaying a polyhedron and a variety of parameters and their values are displayed in two windows to observe while interacting with the 10 plane explosion-implosion computer model, Figure 7b. One window gives the user a view of polyhedra continuously changing shape and rotating to change the view. Some parameters change value continuously while other parameters change value discretely. The Modern Computational Geometry Visualizer window and the Controls window Figure 7b displays values and provide an opportunity to change values. The P:5, Q:2, and R:3, are the symmetry parameters for the vertex, edge, and face symmetries for the icosahedral family of polyhedra.. The p:1, q:1, and r:1, are 3 parameters for the 3 plane model [1]. The r:1, t:1, d:1, and s:1 are the 4 parameters for the 10 plane model. The zoom:value of 294.73%, provides the user with the relative size of the polyhedron being displayed; when the zoom factor is the same for two different displays the two polyhedron are in the same size relative to each other. In general, the 10 plane polyhedra are larger than the 3 plane polyhedra so that it needs a larger zoom factor to be seen in the display window. A mouse is used to either zoom the view in or out with the wheel, or rotate the displayed polyhedron for a different view by moving the mouse in the Visualizer window as if rotating a roller ball.
7.2. Controls. The Controls window provides handles to interact with the 10 plane explosion-implosion computer model. The Camera Rotation continuously rotates the polyhedron, where the Stop Rotation ends this continuous rotation. The Alternate Colors uses two tints of a hue to accentuate the faces of two adjacent fundamental regions, Figure 8a. Polyhedra are considerably different in size because of the edge length of the different polyhedra. The Force Zoom control helps the user see a full polyhedron quickly. The Fund Region control displays only the faces of the fundamental region, Figure 8b, where the Fixed control displays the fundamental region with the lower left corner of the green face is in the center of the screen and the bottom edge and the right edge are parallel to the sides of the screen, Figure 8b. The input for symmetry parameters of the 10 plane model can be changed from \( P \) being 3 for tetrahedral, to 4 for octahedral, and to 5 for isocahedral families of polyhedra. The \( p, q, \) and \( r \) are the coordinates of a point in a unit edge cube for the 3 plane model. The \( \text{reg}, \text{tr}, \text{du}, \) and \( \text{st} \) are the coordinates of a point in a unit edge hypercube for the 10 plane model. Plane count allows the user to choose between the 3 plane model and the 10 plane model.

7.3. Colors. The 10 faces were organized into 4 sets for coloring. A first set of 3 faces was the primary colors of red, green and blue for a regular polyhedron, Figure 4a. A second set of 3
faces were the secondary colors of yellow, cyan, and magenta for a stellate polyhedron, Figure 4d. The face for a dual polyhedron, Figure 4c, was white and faces for a truncate polyhedron, Figure 4b were red, green, blue, and white. The final 3 faces were pink, lime, and aqua as blends between red and white, green and white, and blue and white Figure 8b. The hue of each color has been adjusted, attempting to produce displays, where the faces appear to be adjacent to each other in their 3-dimensional viewing space.

7.4 Plotting. The Face Plotting option in the Control window displays faces that could be cut out to build a physical model of a polyhedron from the 10 plane model, Figure 9.

8. Animation, Documentary, Interaction.

8.1. Animation. While a staff scientist at NYIT Computer Graphics Laboratory from 1980 to 1990, a 3 plane computer model for an explosion-implosion polyhedral transformation was animated. A range of shape changes for this computer model was presented in 10 minutes. There was a positive response to the first showing of this animation at a conference of geometers at Smith College in 1989. It was very exciting for me to have Coxeter view the animation as part of a solution for the computer model. The algebra for this polyhedral geometric model started with his writings [4]. After viewing the animation he said, “This work is for a next generation of geometers”. The animation introduced viewers to a dynamically changing shape displayed for the explosion-implosion of a 3 plane computer model.

8.2. Documentary. In the fall of 2006 an interactive program was started that would further assist in the understanding of this 3 plane model by utilizing an interactive display instead of an animation. We started with old C code and knowledge of C++ code, OpenGL and GLUT to produce an interactive program exhibiting a solution to a 3 plane explosion-implosion computer model. At first we aspired to tell the same story that was told in the animation. We changed our thinking to fit an interactive display to be video taped. A documentary of Will interacting with the code was produced. This documentary can be seen on a DVD and is considerably different from the animation, but it displays a similar range of shapes that appear in the animation, however it produced a considerably higher level of energy utilizing an interactive display.

8.3. Interaction. The 3 plane model has been extended to a 10 plane model. While at NYIT in 1989, while visiting with Pat Hanrahan, when he was faculty at Princeton, an initial result was
obtained [5]. This initial result has been extended into an interactive program for the 10 plane model. As the 10 plane model work developed values for parameters were displayed as well as the opportunity to interactively change values for parameters. Viewing this interactive explosion-implosion polyhedral transformation has proven to be considerably more stimulating. As the complexity of the polyhedron grows so grows the stimulation from viewing the changes of shape. Initially animation was helpful in telling a story, eventually interactivity became a more powerful experience to tell the story of a dynamic phenomenon of polyhedra changing shape.

9. Conclusion

A 10 plane explosion-implosion polyhedral transformation was implemented to provide an interactive display of polyhedra that are well-behaved structures in a 3-dimensional space. Each polyhedron is referenced by a 4-dimensional point from a unit edge hypercube. Lalvani’s explosion-implosion polyhedral transformation was conceptually clear from his dissertation and his exhibit. A computer model was derived for this concept by writing C code, in the late 1980’s, producing computer graphics animation as a result in 1989. More recently, Will Hawkins and I have been able to update the 1980’s code for an interactive display.

Animation, documentation, and interaction are different media for telling a story. When attempting to tell a similar story the different medium accentuate their strengths. Given a choice, interaction is the media that generates the highest level of energy. It does not mean that the others media are less valuable, only that they are different in that the technology to support each is very different. Animation, documentation, and interaction each have a significant value and all 3 help students to better understanding a 3-dimensional dynamic computer aided geometric model.

Acknowledgement

I would like to thank my spouse, Deborah, for her proof reading drafts of this paper.

References