Designing Symmetric Peano Curve Tiling Patterns with Escher-esque Foreground/Background Ambiguity

Douglas McKenna • Mathemæsthetics, Inc.
1140 Linden Ave. • Boulder • Colorado • 80304
doug@mathemæsthetics.com

Abstract

By generalizing Peano's original space-filling curve construction, one can design out of simple square tiles modular, two-color tiling patterns whose foreground and background are "self-negative" under 180° rotation. The resultant abstract geometric tiling patterns have an "Escher-esque" quality due to the attendant ambiguity that arises when the visual system cannot tell which connected area is foreground and which is background.

Introduction

I have long been interested in exploring the combinatorics of space-filling curve constructions for the purpose of creating elegant and eye-catching geometric designs. Figure 1 is a two-foot wide drawing I made in 1984 (in pre-PostScript® days) with a pen plotter using India ink.

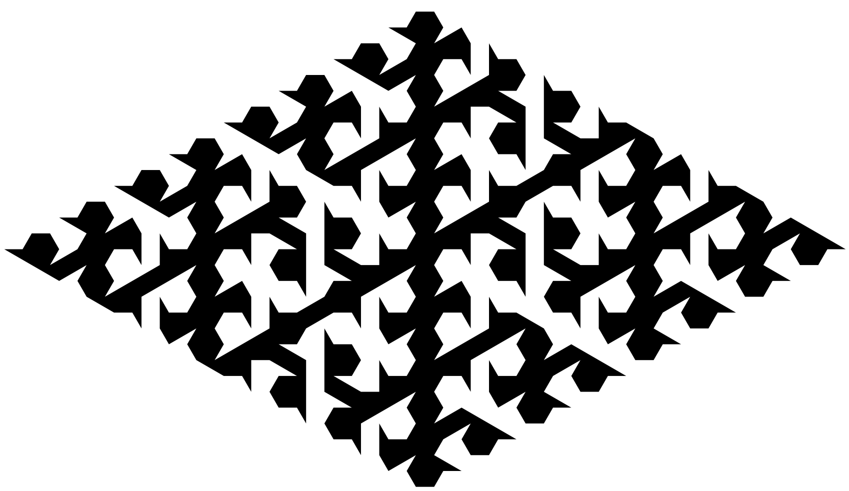

Figure 1: *A triangle-based, space-filling curve design with locally identical foreground and background.*

A response that Figure 1 regularly elicits from viewers is that it is "reminiscent" of Escher. This is because the abstract design has a sweet visual property—the connected foreground and the connected background, at least in the local interior, cannot be distinguished on the basis of shape. This ambiguity is what attracts the eye. Is it white on black, or black on white? The confusion plays tricks with our visual system's desire to integrate shapes into meaning. This property is the basis of a variety of Escher's playful, tiling-based works. But it is also a property related to recursive, space-filling curve constructions whose initial stages are illustrated as self-avoiding paths that "thread" underlying tilings. So I've gone back to the original Peano Curve construction, to generalize, explore, enumerate, and make a few aesthetic choices.

Peano's construction is illustrated in one of two ways, as shown in Figure 2. The first is based on his original (non-illustrated) paper [1], as described in Sagan's book [2] and earlier (e.g., [3]). The second is from the public Wikipedia website [4]. With a surfeit of vertical lines, the latter seems more visually elegant, although the former is the original construction.

The path that threads the nine sub-squares has a central point of rotational symmetry, which is maintained after reduced copies of the pattern are placed in each sub-square and then connected across sub-square boundaries into one longer path (second column, ignoring diagonal lines). Thus, when the areas on either side of the self-avoiding path are colored oppositely, the resulting design is equivalent to a "self-negative" tiling under 180° rotation: the boundary between black and white remains the same, but black and white tiles swap.

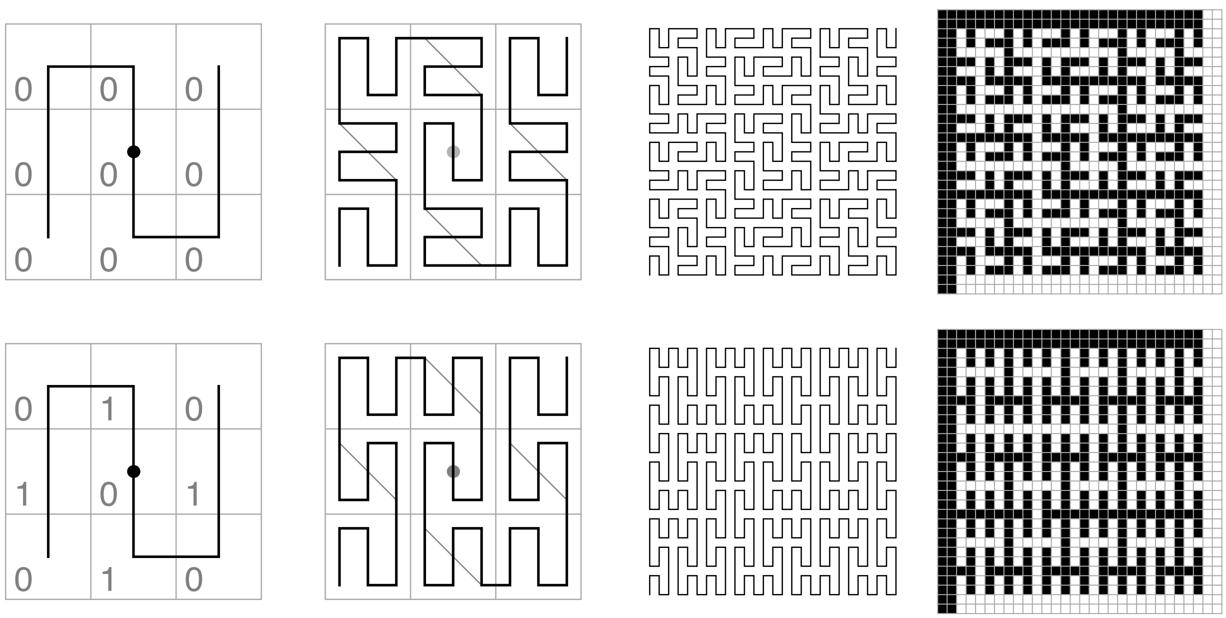

Figure 2: *Two Peano curve constructions and related tile designs. A* 1 *bit causes mirroring at the next stage.*

I've added diagonal lines in four of the nine sub-squares (those marked with a 1 on the left) to illustrate the difference between these two constructions. When each of the four internal patterns is mirrored across its respective diagonal line within its sub-square, one construction converts to the other without affecting the self-avoidance or connectivity of the path. Diagonal mirror lines added to the remaining five sub-squares would be in the opposite diagonal orientation. Because this extra degree of mirroring freedom occurs independently in all 9 sub-squares of the 3×3 sub-division, there are $2^9 = 512$ possible ways to construct these simplest Peano curves. The differences only become apparent after the second recursive stage. Each construction is specified by the original generator's shape, and a 9-bit binary value. When the bits are palindromic (e.g., 0101-0-1010), rotational symmetry is maintained. Thus, all self-negative possibilities are dependent upon only the first five bits (the remaining four mirror the first four), and so there are $2^5 = 32$ possible symmetric constructions. Figure 2 shows two of them; Figure 3 shows the remaining 30 as self-negative tile designs. Of these, Figure 4 shows several visually interesting ones.

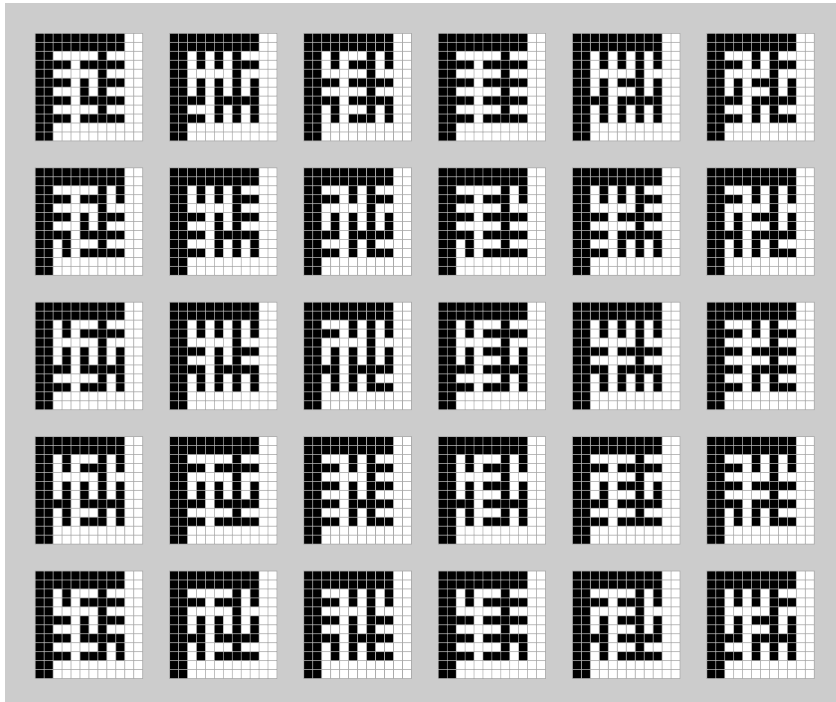

Figure 3: *The remaining 30 out of 32 self-negative, Peano Curve constructions (at stage 2), as tiling patterns.*

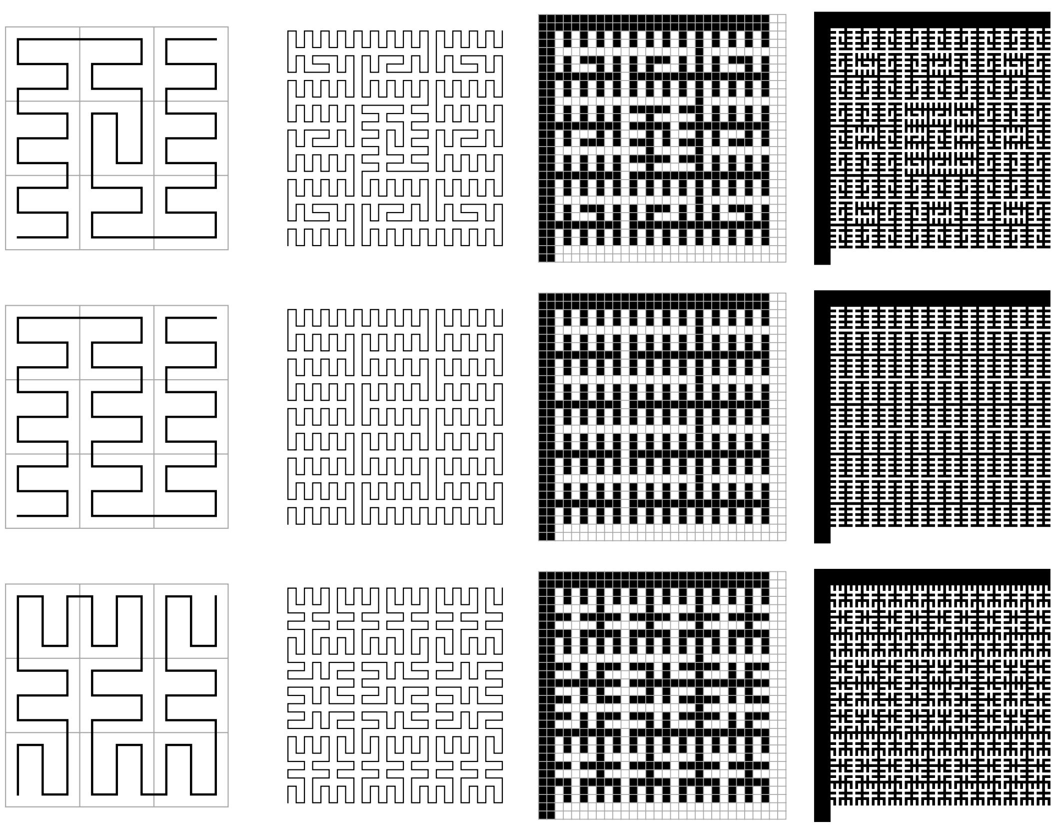

Figure 4: *Several visually interesting tilings chosen from among the remaining 30 constructions.*

Higher Order Subdivisions Offer Richer and More Numerous Ornamental Possibilities

There is only one distinct, self-avoiding way to traverse the 9 sub-squares of a 3×3 subdivision from lower left to upper right. Consequently there is a level of shape homogeneity to all of the previous Peano Curve patterns that works against them as aesthetic objects. A higher resolution palette, however, opens up more combinatorial possibilities. This makes for a richer space from which to make choices based on visual, aesthetic, or other criteria.

So let's generalize to an $n \times n$ subdivision, starting with $n = 5$. We can't use an even n, because the self-avoiding paths these constructions create must pass through the centers of sub-squares *as well as* the original square's point of rotational symmetry. When n is even, however, there is no single central sub-square in the subdivision that contains that symmetry point.

After coloring each side of the path black or white, the result for the kth recursive construction stage is an $(n^k - 1) \times (n^k - 1)$ pattern (not counting any extra surrounding margins of black or white tiles). So on the order 5 subdivision, Figure 5 shows the 18 possible 4×4 tilings (with added 2-tile-wide borders) that a computer enumeration finds. Four (boxed in the top row) are rotationally self-negative, and another three pairs are mutual rotational negatives of each other. The four patterns illustrate three flavors of self-negative form: spirals, zig-zags, and interfingering.

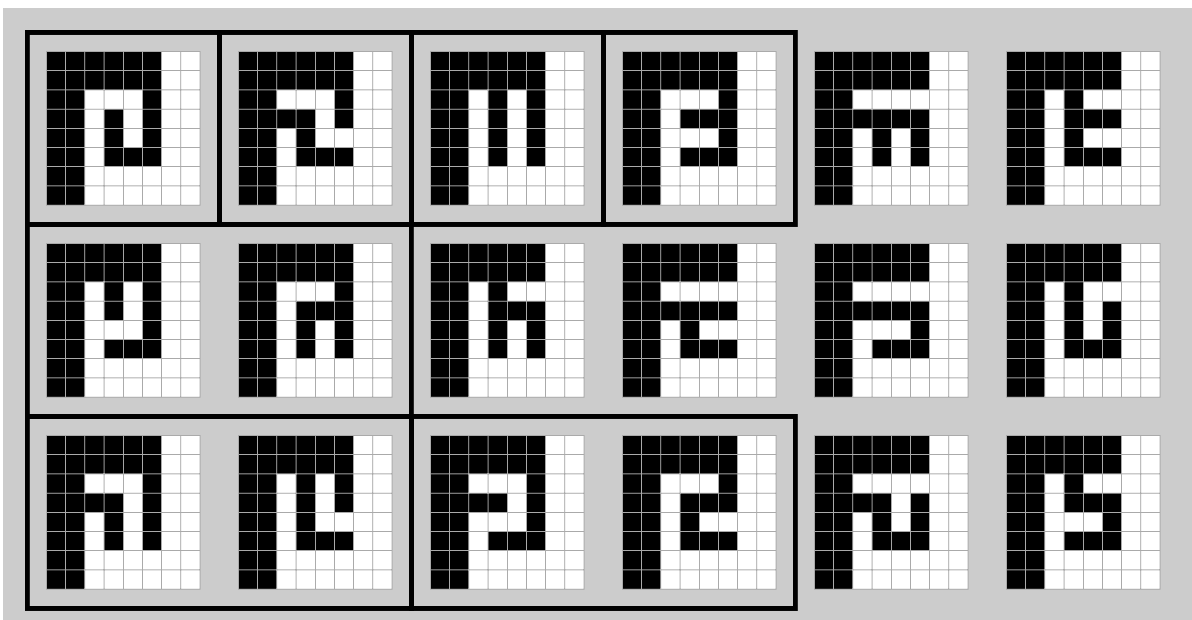

Figure 5: *All order* 5 *space-filling curve generators, rendered as* 4×4 *tilings with 2-tile borders.*

Any one of the rotationally symmetric generator shapes can be combined with 25 bits that specify the mirroring of sequential sub-squares at the second and later recursive stages of the space-filling construction. When palindromic, only the first $\lceil 25/2 \rceil = 13$ bits are independent of each other, so there are $4 \times 2^{13} = 4 \times 8192 = 32768$ possible symmetric constructions, 8192 for each of the four self-negative generators. This constitutes a sufficiently rich enough space that it becomes possible to *design*, not just *discover*, space-filling constructions and their attendant tiling patterns. Figures 6 and 7 show four elegant designs that I find particularly pleasing.

On the order 7 subdivision, a computer enumeration finds 3364 generator shapes, of which the 60 rotationally self-negative ones are shown in Figure 8. Each of them has $2^{\lceil 49/2 \rceil = 25} = 33,554,431$ possible symmetric mirroring variations. On the order 9 subdivision, there are almost 6.5 million

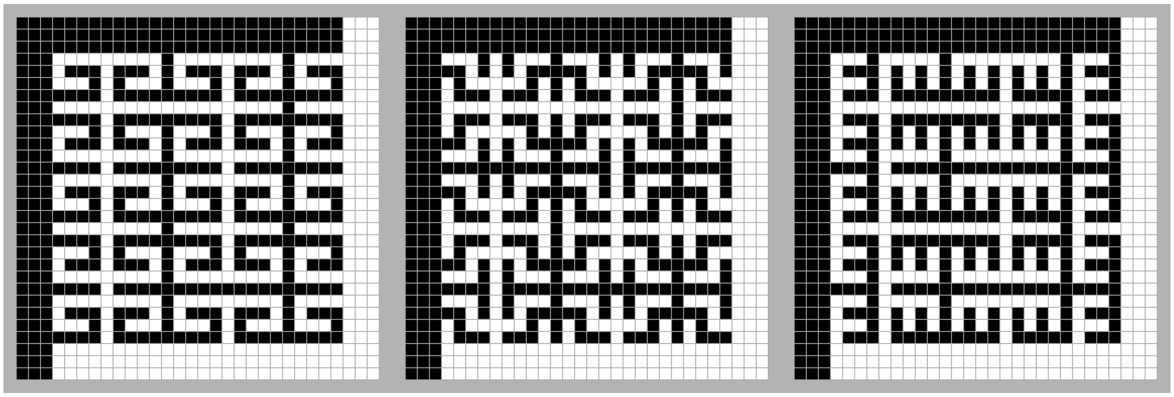

Figure 6: *Three self-negative tile designs, with 3-tile borders.*

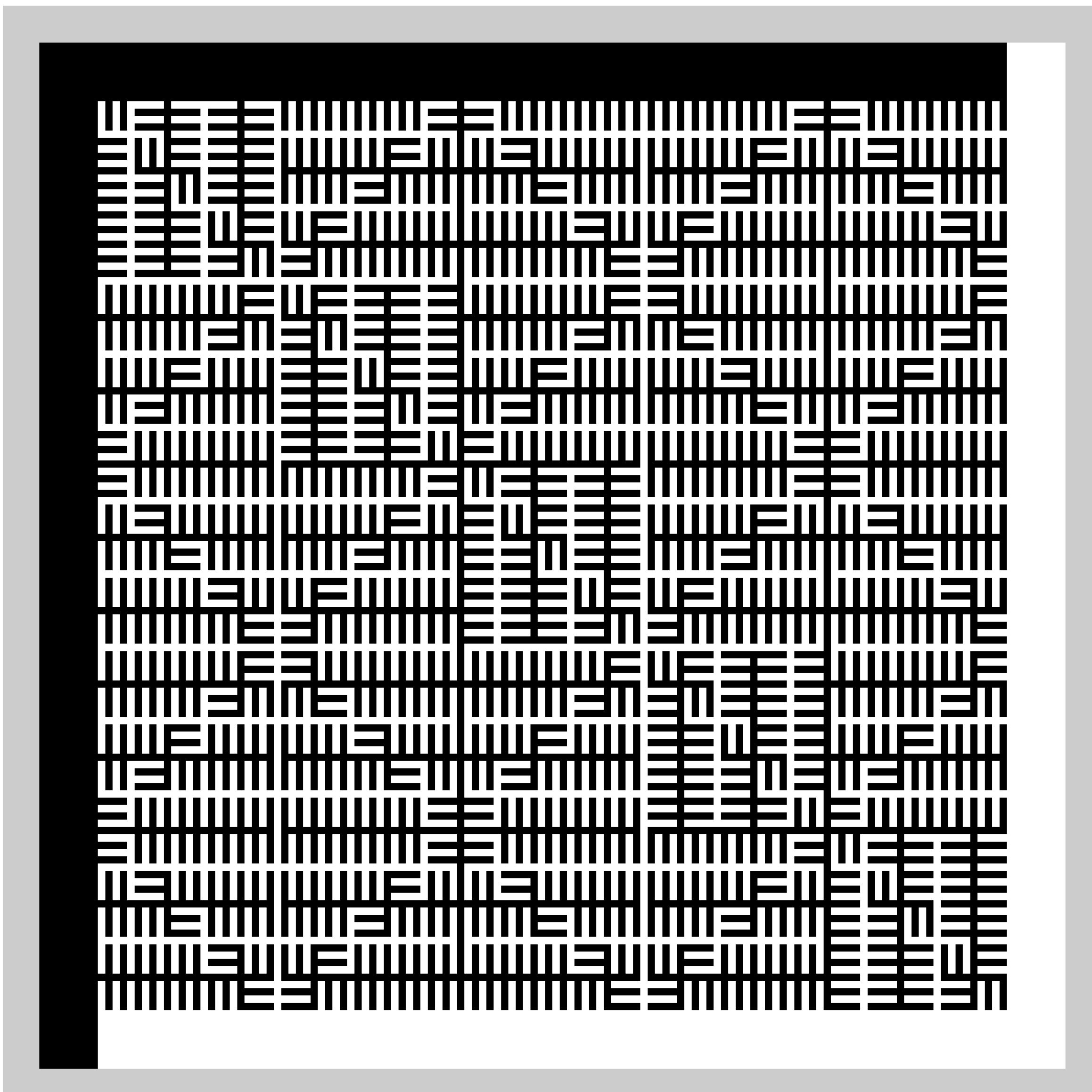

Figure 7: *"Criss Cross", © 2008 Douglas M. McKenna. (Grout lines have been left out.)*

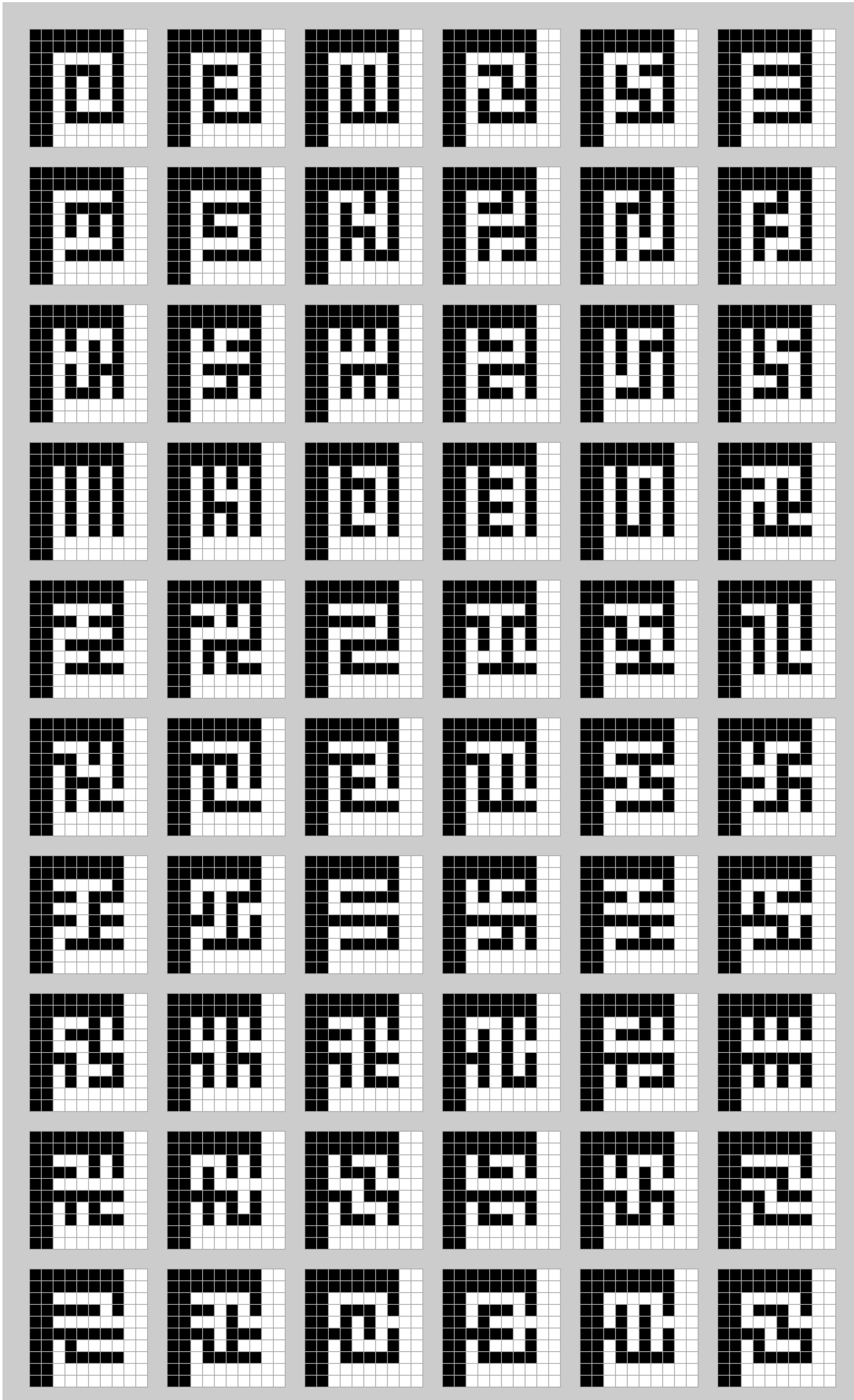

Figure 8: *The 60 order 7 generators, with 2-tile-wide borders, that are self-negative under 180° rotation.*

generators. 2640 of them are self-negative, and each of those has 2^{41} (over 2 trillion) possible symmetric mirroring possibilities. Figure 9 shows a visually intriguing example that uses alternating mirroring to organize the space-filling curve's connected sub-square patterns in a way that emphasizes local symmetries. This piece is both locally and globally self-negative. Close inspection shows that Figure 9 looks periodic, but is not. Note that the border between black and white is a Hamiltonian path: when the edges of the $(9^2-1)\times(9^2-1)=1600$ sub-squares are treated as a grid graph, every vertex is visited exactly once as the path traverses from lower-left to upper-right.

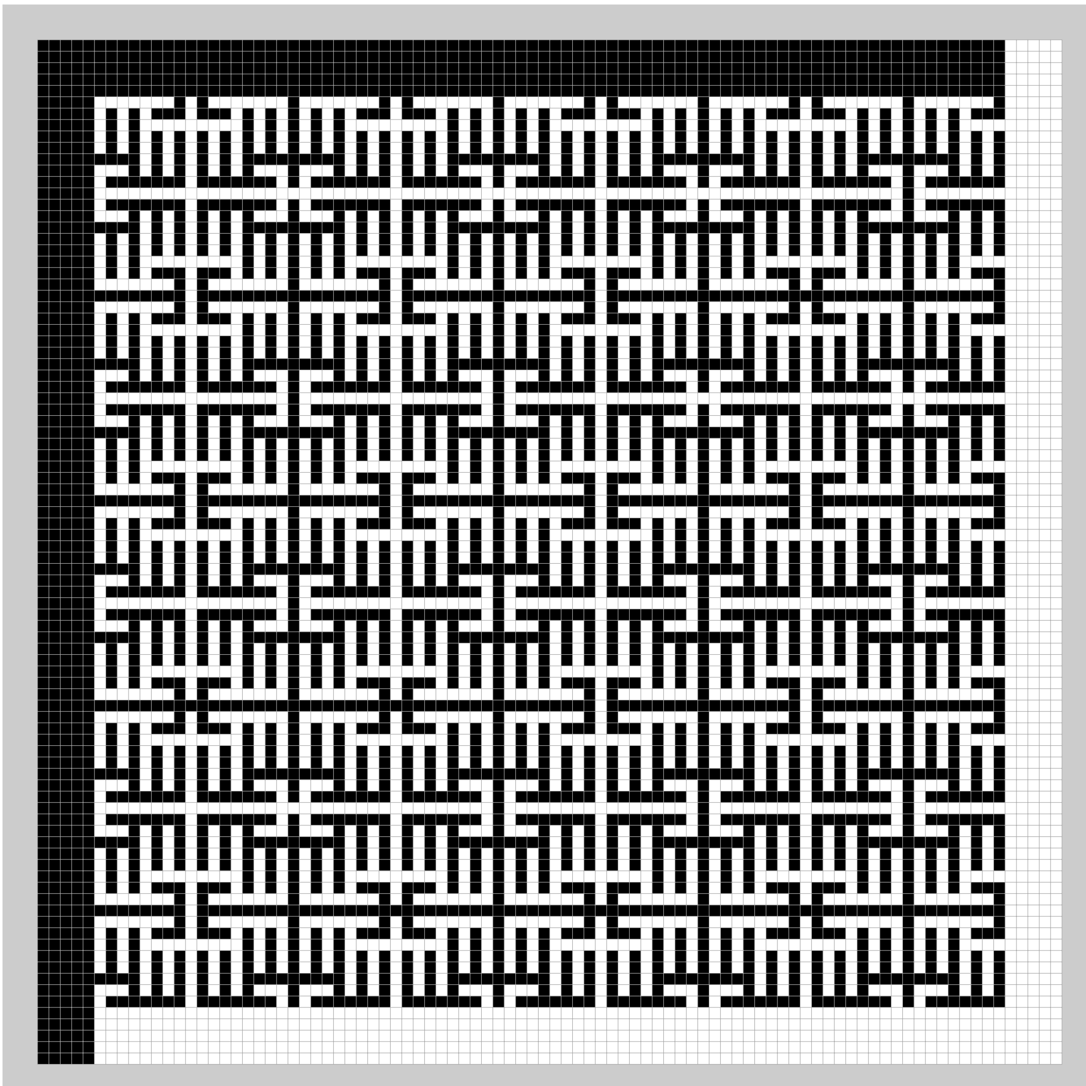

Figure 9: *"Synaptica" © 2008 Douglas M. McKenna.*

Other Ornamental Possibilities

The foregoing designs have all been strictly recursive. But recursion, while elegant, is not necessary. In particular, *any* zig-zag path traveling along the diagonals of a sequence of edge-adjacent squares serves as a template to place and connect these tiling patterns and their mirrors.

One can pick and choose among generators based on a variety of visual or design considerations: diagonal elements, horizontal vs. vertical imbalance, letterform-like patterns, etc. Indeed, it seems rather fitting that the most common interfingering motif among these self-negative patterns is one reminiscent of the block letter 'E', for Escher. M. C. Escher signed his art with "M C E" drawn as if each letter were a square tile. He might have enjoyed something along the lines of Figure 10. The center is simply three of these self-negative, order 7 Peano curve generators, connected and mirrored appropriately. Three others form "M C E"–reminiscent ornamentation surrounding the central initials.

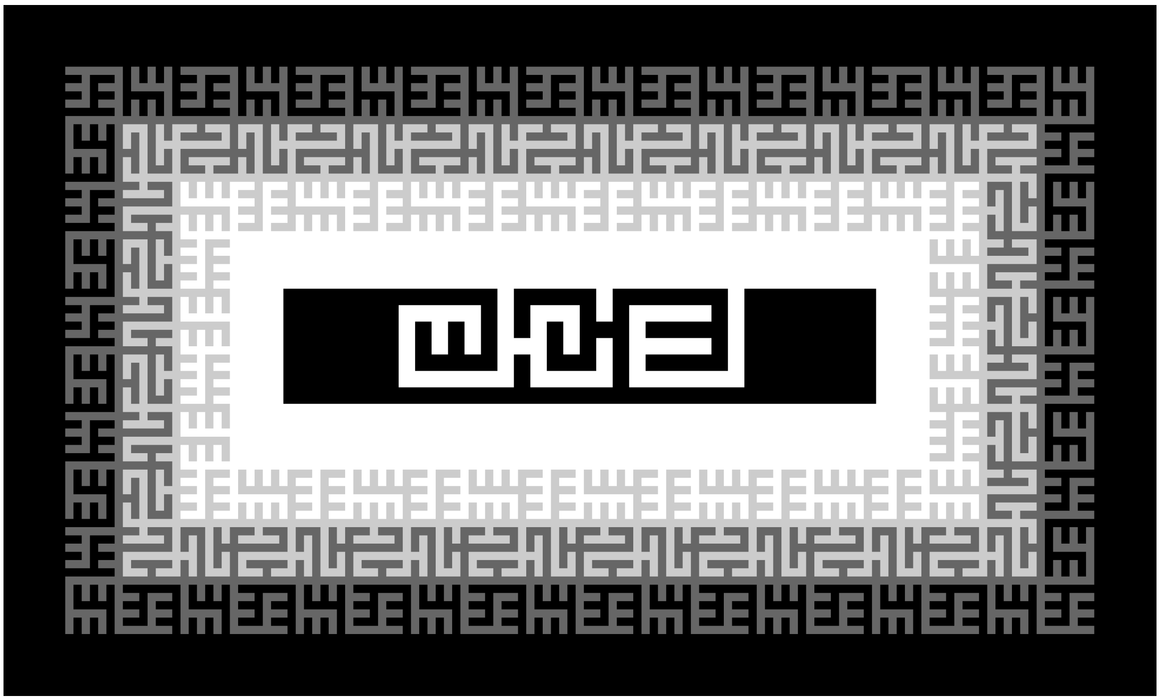

Figure 10: *A sample "M C E" carpet (or tiling), designed using self-negative, order 7 tiling patterns.*

Conclusion

By generalizing the space-filling Peano Curve construction through careful mirroring and higher order subdivisions, one can create myriad square tile designs with rotationally self-negative patterns, within which visually fascinating, foreground/background ambiguities arise. Elegant tile designs can be constructed modularly, making them easy to lay out and build, using stock square tiles. And one can also leverage the visual ambiguity of these patterns to create border motifs, or other forms of eye-pleasing ornamentation, of which the world is in constant need.

References

[1] Peano, G., "Sur Une Courbe Qui Remplit Toute Une Aire Plaine", *Math. Annalen* **36** (1890), pp. 157-160.

[2] Sagan, H., *Space-Filling Curves*, Springer-Verlag (1994), ch. 1–3.

[3] Mandelbrot, B. B., *The Fractal Geometry of Nature*, W. H. Freeman (1982), p. 62.

[4] See <http://en.wikipedia.org/wiki/Space_filling_curve> (last accessed 02/04/08)